

Zeta-values of one-dimensional arithmetic schemes at strictly negative integers

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Abstract

Let X be an arithmetic scheme (i.e., separated, of finite type over $\text{Spec } \mathbb{Z}$) of Krull dimension 1. For the associated zeta function $\zeta(X, s)$, we write down a formula for the special value at $s = n < 0$ in terms of the étale motivic cohomology of X and a regulator. We prove it in the case when for each generic point $\eta \in X$ with $\text{char } \kappa(\eta) = 0$, the extension $\kappa(\eta)/\mathbb{Q}$ is abelian. We conjecture that the formula holds for any one-dimensional arithmetic scheme.

This is a consequence of the Weil-étale formalism developed by the author in [2] and [3], following the work of Flach and Morin [8]. We also calculate the Weil-étale cohomology of one-dimensional arithmetic schemes and show that our special value formula is a particular case of the main conjecture from [3].

1 Introduction

Let X be an **arithmetic scheme**, by which we mean in this text that it is separated and of finite type over $\text{Spec } \mathbb{Z}$. The **zeta function** associated to X (see, e.g. [36]) is given by

$$\zeta(X, s) := \prod_{\substack{x \in X \\ \text{closed pt.}}} \frac{1}{1 - N(x)^{-s}},$$

where the norm of a closed point $x \in X$ is the size of the corresponding residue field:

$$N(x) := |\kappa(x)| := |\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}|$$

The above product converges for $\text{Re } s > \dim X$ and is supposed to have a meromorphic continuation to the whole complex plane. Although the latter is a wide-open conjecture in general, it is well-known for one-dimensional schemes, which is the case of interest in this article.

If $\zeta(X, s)$ admits a meromorphic continuation around $s = n$, we denote by

$$d_n := \text{ord}_{s=n} \zeta(X, s) \tag{1}$$

the **vanishing order** of $\zeta(X, s)$ at $s = n$. The corresponding **special value** of $\zeta(X, s)$ at $s = n$ is defined as the leading nonzero coefficient of the Taylor expansion:

$$\zeta^*(X, n) := \lim_{s \rightarrow n} (s - n)^{-d_n} \zeta(X, s).$$

Since the 19th century, many formulas (both conjectural and unconditional) have been proposed to interpret the numbers $\zeta^*(X, n)$ in terms of geometric and algebraic invariants attached to X . A primordial example is Dirichlet's **analytic class number formula**. For a number field F/\mathbb{Q} , we denote by \mathcal{O}_F the corresponding ring of integers. Then

$$\zeta_F(s) := \zeta(\mathrm{Spec} \mathcal{O}_F, s)$$

is the **Dedekind zeta function** attached to F . From the well-known functional equation for $\zeta_F(s)$, it is easy to see that it has a zero at $s = 0$ of order $r_1 + r_2 - 1$, where r_1 (resp. $2r_2$) is the number of real embeddings $F \hookrightarrow \mathbb{R}$ (resp. complex embeddings $F \hookrightarrow \mathbb{C}$). The corresponding special value at $s = 0$ is given by

$$\zeta_F^*(0) = -\frac{h_F}{\omega_F} R_F, \tag{2}$$

where $h_F = |\mathrm{Pic}(\mathcal{O}_F)|$ is the class number, $\omega_F = |(\mathcal{O}_F)_{tors}^\times|$ is the number of roots of unity in F , and $R_F \in \mathbb{R}$ is the regulator. See, e.g., [7, Chapter 5, §1] or [34, §VII.5].

The question naturally arises whether there are formulas similar to (2) for $s = n \in \mathbb{Z}$ other than $s = 0$ (or $s = 1$, which is related to $s = 0$ via the functional equation). To do this, one must find a suitable generalization for the numbers h_F, ω_F, R_F . Many special value conjectures of varying generality go back to this question.

Lichtenbaum proposed formulas in terms of algebraic K -theory in his pioneering work [27]. Later these were also reformulated in terms of p -adic cohomology $H^i(\mathrm{Spec} \mathcal{O}_F[1/p]_{\acute{e}t}, \mathbb{Z}_p(n))$ for $i = 1, 2$ and all primes p ; the corresponding formula is known as the **cohomological Lichtenbaum conjecture**; see, for example, [17, §1.7] for the statement and a proof for abelian number fields F/\mathbb{Q} . We will not go into details here, since it is more convenient for us to use motivic cohomology instead of working with p -adic cohomology for varying p .

A suitable generalization of R_F are the **higher regulators** considered since the work of Borel [6] and later by Beilinson [1].

We do not attempt to give an adequate historical survey of the subject or to write down all the conjectured formulas; the interested reader may consult, e.g., [25, 16, 21].

Later, Lichtenbaum proposed a new research program known as **Weil-étale cohomology**; see [28, 29, 30, 31]. It suggests that for an arithmetic scheme X the special value of $\zeta(X, s)$ at $s = n \in \mathbb{Z}$ can be expressed in terms of the Weil-étale cohomology, which is a suitable modification of the étale motivic cohomology of X . Flach and Morin in [8] gave a construction of Weil-étale

cohomology groups $H_{W,c}^i(X, \mathbb{Z}(n))$ for a proper and regular arithmetic scheme X , and stated a precise conjectural relation of $H_{W,c}^i(X, \mathbb{Z}(n))$ to the special value $\zeta^*(X, n)$.

In [8, §5.8.3] they write down an explicit formula for the case of $X = \text{Spec } \mathcal{O}_F$. For $n \leq 0$ and in terms of cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$, it reads

$$\zeta_F^*(n) = \pm \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}|} R_{F,n} \quad \text{for } n \leq 0. \quad (3)$$

The definition of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is reviewed below. The regulator $R_{F,n} = R_{\text{Spec } \mathcal{O}_F, n}$ is defined in §6.

By [8, Proposition 5.35], formula (3) holds unconditionally for abelian number fields F/\mathbb{Q} , via a reduction to the **Tamagawa number conjecture** of Bloch–Kato–Fontaine–Perrin-Riou.

In particular, if we take $n = 0$, then $\mathbb{Z}^c(0) \cong \mathbb{G}_m[1]$, and $R_{F,0}$ is the usual Dirichlet regulator, so (3) becomes the classical formula (2):

$$\zeta_F^*(0) = \pm \frac{|H^1(\text{Spec } \mathcal{O}_F, \acute{e}t, \mathbb{G}_m)|}{|H^0(\text{Spec } \mathcal{O}_F, \acute{e}t, \mathbb{G}_m)_{tors}|} R_F = \pm \frac{|\text{Pic}(\mathcal{O}_F)|}{|(\mathcal{O}_F)_{tors}^\times|} R_F,$$

We also mention that Flach and Morin have a similar special value formula for $n > 0$, which includes a correction factor $C(X, n) \in \mathbb{Q}$. In this text we will say nothing about the case of $n > 0$; the reader can consult [8] for more details, and also the subsequent papers [9, 10, 33] which shed light on the nature of the correction factor $C(X, n)$.

For $n < 0$, the author in [2] and [3] extended the work of Flach and Morin [8] to an arbitrary arithmetic scheme X (thus removing the assumption that X is proper or regular). In this text, we would like to work out explicitly the corresponding special value formula for one-dimensional arithmetic schemes.

To state the main result, it is useful to introduce the following terminology.

DEFINITION 1.1. We say that a one-dimensional arithmetic scheme X is **abelian** if each generic point $\eta \in X$ with $\text{char } \kappa(\eta) = 0$ corresponds to an abelian extension $\kappa(\eta)/\mathbb{Q}$.

If X lives in positive characteristic, then it is trivially abelian. The term “abelian” is ad hoc and was suggested by analogy with the notion of **abelian number fields**. Hopefully there is no confusion with the “abelian schemes” that are generalizations of abelian varieties.

Our goal is to prove the following result.

THEOREM 1.2. *For an abelian one-dimensional arithmetic scheme X , the special value of $\zeta(X, s)$ at $s = n < 0$ is given by*

$$\zeta^*(X, n) = \pm 2^\delta \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} R_{X,n}. \quad (4)$$

Here

- $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ the étale motivic cohomology from [13];
- the correction factor 2^δ is given by

$$\delta = \delta_{X,n} = \begin{cases} r_1, & n \text{ even,} \\ 0, & n \text{ odd,} \end{cases} \quad (5)$$

where $r_1 = |X(\mathbb{R})|$ is the number of real places of X ,

- $R_{X,n}$ is a positive real number defined via a regulator map in §6.

We further conjecture that formula (4) holds for all one-dimensional arithmetic schemes, not necessarily abelian. This is equivalent to the Tamagawa number conjecture for non-abelian number fields (see Remark 7.4).

We give two proofs of (4): first a direct argument in §7 and then an argument in terms of Weil-étale cohomology in §9. In fact, we note that the special value formula is the same as the conjecture $\mathbf{C}(X, n)$ formulated in [3], which is specialized to one-dimensional X and spelled out explicitly.

The purpose of this text is twofold. First, we establish a new special value formula, which generalizes several formulas found in the literature. Second, we review the construction of Weil-étale cohomology $H_{W,c}^i(X, \mathbb{Z}(n))$ from [2] and the special value conjecture from [3] and explain it in the case of one-dimensional schemes. It is not very surprising that a special value formula like (4) exists, but the right cohomological invariants to state it have been suggested by the Weil-étale framework.

This text was inspired in part by the work of Jordan and Poonen [19], which deals with a formula for $\zeta^*(X, 1)$, where X is an affine reduced one-dimensional arithmetic scheme. The affine and reduced constraint does not appear in our case because work with different invariants. Since $\zeta(X, s) = \zeta(X_{red}, s)$, the “right” invariants should not distinguish between X and X_{red} , and motivic cohomology satisfies this property.

Notation and conventions

Abelian groups. For an abelian group A , we denote

$$\begin{aligned} A^D &:= \text{Hom}(A, \mathbb{Q}/\mathbb{Z}), \\ A^* &:= \text{Hom}(A, \mathbb{Z}). \end{aligned}$$

There is an exact sequence

$$0 \rightarrow A^* \rightarrow \mathrm{Hom}(A, \mathbb{Q}) \rightarrow A^D \rightarrow (A_{tors})^D \rightarrow 0 \quad (6)$$

Note that for a finite rank group A , the \mathbb{Z} -dual A^* is free and has the same rank. If A is finite, then there is a (non-canonical) isomorphism with the \mathbb{Q}/\mathbb{Z} -dual $A \cong A^D$, and in particular $|A^D| = |A|$.

Schemes. In this text, X always denotes a **one-dimensional arithmetic scheme**, i.e., a separated scheme of finite type $X \rightarrow \mathrm{Spec} \mathbb{Z}$ of Krull dimension 1.

We remark that the restriction that X is abelian (Definition 1.1) is needed only for the proofs of Theorem 1.2 in §7 and §9. Our calculations in §§3, 4, 5, 6, 8 work for any one-dimensional arithmetic scheme X .

Weights. In this text, n always stands for a fixed, *strictly negative* integer.

Motivic cohomology. We will work with a version of étale motivic cohomology defined in terms of **Bloch's cycle complexes**. These were introduced by Bloch in [4] for varieties over fields, and for the version over $\mathrm{Spec} \mathbb{Z}$ see [11, 12].

In short, we let $\Delta^i = \mathrm{Spec} \mathbb{Z}[t_0, \dots, t_i]/(1 - \sum_i t_i)$ be the algebraic simplex. Denote by $z_n(X, i)$ the group freely generated by algebraic cycles $Z \subset X \times \Delta^i$ of dimension $n + i$ that intersect the faces properly. For $n < 0$ we consider the complex of sheaves on $X_{\acute{e}t}$

$$\mathbb{Z}^c(n) := z_n(-, -\bullet)[2n].$$

The corresponding (hyper)cohomology

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) := H^i(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)))$$

is what we will call in this text **(étale) motivic cohomology**. For a proper regular arithmetic scheme X of pure dimension d we have

$$\mathbb{Z}^c(n) \cong \mathbb{Z}(d - n)[2d], \quad (7)$$

where $\mathbb{Z}(m)$ is the other motivic complex that usually appears in the literature; see [11, 12] for the definition. To avoid any confusion, all our calculations will be in terms of $\mathbb{Z}^c(n)$.

By [13, Corollary 7.2], the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ satisfy the **localization property**: if $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is its closed complement, then there is a distinguished triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1],$$

giving a long exact sequence

$$\cdots \rightarrow H^i(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^i(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^{i+1}(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow \cdots \quad (8)$$

This means that $H^i(-, \mathbb{Z}^c(n))$ behaves like (motivic) Borel–Moore homology.

At the level of zeta functions, the localization property corresponds to the identity

$$\zeta(X, s) = \zeta(Z, s) \zeta(U, s).$$

For more results on $\mathbb{Z}^c(n)$, we refer the reader to [13].

In general, the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are very hard to compute. However, they are quite well understood for one-dimensional arithmetic schemes X ; see §5 below.

Outline of the paper

In §2 we prove a dévissage lemma that shows how a property that holds for curves over finite fields and for number rings can be generalized to any one-dimensional arithmetic scheme. It is an elementary argument, isolated to avoid repeating the same reasoning in several proofs.

In §3 we calculate the vanishing order of $\zeta(X, s)$ at $s = n < 0$. Then in §4 we calculate the $G_{\mathbb{R}}$ -equivariant cohomology groups of the finite discrete space of complex points $X(\mathbb{C})$. In §5 we put together various well-known results to describe the motivic cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$. In §6 we define the regulator that appears in the special value formula.

Our first “elementary” proof of the main result is given in §7. Then §8 is devoted to a calculation of the Weil-étale cohomology groups $H_{W,c}^i(X, \mathbb{Z}(n))$ from [2] for one-dimensional X , which we consider an interesting result on its own. We use these calculations in §9 to formulate explicitly the conjecture $\mathbf{C}(X, n)$ from [3], again for one-dimensional X . This is a second, more conceptual proof of the main result, and it explains how we arrived at (4) in the first place.

Finally, we conclude in §10 with a couple of examples showing how our special value formula works.

Acknowledgments

I am grateful to Baptiste Morin for various discussions that led to this text.

2 Dévissage lemma for one-dimensional schemes

The main idea of this paper is to consider a property that holds for spectra of number rings $X = \text{Spec } \mathcal{O}_F$ and curves over finite fields X/\mathbb{F}_q , and then generalize it formally to any one-dimensional arithmetic scheme. To this end, in this section we isolate a dévissage argument which will be used repeatedly in the rest of the paper.

LEMMA 2.1. *Let \mathcal{P} be a property of arithmetic schemes of Krull dimension ≤ 1 . Suppose that it satisfies the following compatibilities.*

- a) $\mathcal{P}(X)$ holds if and only if $\mathcal{P}(X_{\text{red}})$ holds.

- b) If $X = \coprod_i X_i$ is a finite disjoint union, then $\mathcal{P}(X)$ is equivalent to the conjunction of $\mathcal{P}(X_i)$ for all i .
- c) If $U \subset X$ is a dense open subscheme, then $\mathcal{P}(X)$ is equivalent to $\mathcal{P}(U)$.

Suppose that

- 0) $\mathcal{P}(\text{Spec } \mathbb{F}_q)$ holds for any finite field \mathbb{F}_q ,
- 1) $\mathcal{P}(X)$ holds for any smooth curve X/\mathbb{F}_q ,
- 2) $\mathcal{P}(\text{Spec } \mathcal{O}_F)$ holds for any number field F/\mathbb{Q} .

Then $\mathcal{P}(X)$ holds for any one-dimensional arithmetic scheme X .

Proof. First suppose that $\dim X = 0$. Then, thanks to a), we can assume that X is reduced, and then $X = \coprod_i \text{Spec } \mathbb{F}_{q,i}$ is a finite disjoint union of spectra of finite fields such that $\mathcal{P}(X)$ holds thanks to 0) and b).

Now consider the case of $\dim X = 1$. Again, we can assume that X is reduced. We take the normalization $\nu: X' \rightarrow X$. This is a birational morphism: there are dense open subschemes $U \subset X$ and $U' \subset X'$ such that $\nu|_{U'}: U' \xrightarrow{\cong} U$ is an isomorphism. Thanks to c), we have

$$\mathcal{P}(X) \iff \mathcal{P}(U) \iff \mathcal{P}(U') \iff \mathcal{P}(X').$$

Therefore, we can assume that X is regular. Now $X = \coprod_i X_i$ is a finite disjoint union of normal integral schemes, so thanks to b), we can assume that X is integral. There are two cases.

- If $X \rightarrow \text{Spec } \mathbb{Z}$ lives over a closed point, then it is a smooth curve over \mathbb{F}_q , and the claim holds thanks to 1).
- If $X \rightarrow \text{Spec } \mathbb{Z}$ is a dominant morphism, consider an open affine neighborhood of the generic point $U \subset X$. Again, $\mathcal{P}(X)$ is equivalent to $\mathcal{P}(U)$, so it suffices to prove the claim for U . We have $U = \text{Spec } \mathcal{O}_{F,S}$ for a number field F/\mathbb{Q} and a finite set of places S , so everything reduces to $\mathcal{P}(\text{Spec } \mathcal{O}_F)$. \square

3 Vanishing order of $\zeta(X, s)$ at $s = n < 0$

DEFINITION 3.1 (Numbers r_1 and r_2). Given a one-dimensional arithmetic scheme X , consider the finite discrete space of complex points

$$X(\mathbb{C}) := \text{Hom}(\text{Spec } \mathbb{C}, X).$$

There is a canonical action of the complex conjugation $G_{\mathbb{R}} := \text{Gal}(\mathbb{C}/\mathbb{R})$ on $X(\mathbb{C})$. The fixed points of this action correspond to the real points $X(\mathbb{R})$, also known as the **real places**. We set $r_1 = |X(\mathbb{R})|$. The non-real places are called **complex places**. They come in conjugate pairs, and we denote their number by $2r_2$.

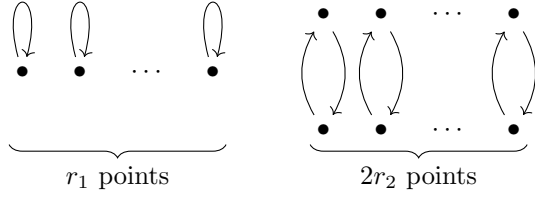


Figure 1: $G_{\mathbb{R}} := \text{Gal}(\mathbb{C}/\mathbb{R})$ acting on $X(\mathbb{C})$

Equivalently, for a number field F/\mathbb{Q} , denote by $r_1(F)$ the number of real embeddings $F \hookrightarrow \mathbb{R}$ and by $r_2(F)$ the number of pairs of complex embeddings $F \hookrightarrow \mathbb{C}$. Then $r_1(F) = r_1$ and $r_2(F) = r_2$ for $X = \text{Spec } \mathcal{O}_F$. In general, for a one-dimensional arithmetic scheme X , we have

$$r_1 = \sum_{\text{char } \kappa(\eta)=0} r_1(\kappa(\eta)),$$

$$r_2 = \sum_{\text{char } \kappa(\eta)=0} r_2(\kappa(\eta)),$$

where the sums are over generic points $\eta \in X$ with residue field $\kappa(\eta)$ of characteristic 0.

PROPOSITION 3.2. *Let X be a one-dimensional arithmetic scheme with r_1 real and $2r_2$ complex places. For $n < 0$, the vanishing order of $\zeta(X, s)$ at $s = n$ is given by*

$$d_n = \text{ord}_{s=n} \zeta(X, s) = \begin{cases} r_1 + r_2, & n \text{ even,} \\ r_2, & n \text{ odd.} \end{cases} \quad (9)$$

Proof. For $X = \text{Spec } \mathcal{O}_F$ the claim is a well-known consequence of the functional equation for the Dedekind zeta function [34, §VII.5]. It also holds for X/\mathbb{F}_q since in this case $\zeta(X, s)$ has no zeros or poles at $s = n < 0$ according to [22, pp. 26–27]. We now proceed using Lemma 2.1.

We have $\zeta(X, s) = \zeta(X_{\text{red}}, s)$ and $r_{1,2}(X) = r_{1,2}(X_{\text{red}})$. If $X = \coprod_i X_i$ is a finite disjoint union, then

$$\text{ord}_{s=n} \zeta(X, s) = \sum_i \text{ord}_{s=n} \zeta(X_i, s),$$

$$r_{1,2}(X) = \sum_i r_{1,2}(X_i),$$

so that the property is compatible with disjoint unions. Finally, if $U \subset X$ is a dense open subscheme, then $Z = X \setminus U$ is a zero-dimensional scheme, and

$$\text{ord}_{s=n} \zeta(X, s) = \text{ord}_{s=n} \zeta(U, s),$$

$$r_{1,2}(X) = r_{1,2}(U),$$

so that the property is compatible with taking dense open subschemes. We conclude that Lemma 2.1 applies. \square

4 $G_{\mathbb{R}}$ -equivariant cohomology of $X(\mathbb{C})$

Viewing $\mathbb{Z}(n) := (2\pi i)^n \mathbb{Z}$ as a constant $G_{\mathbb{R}}$ -equivariant sheaf on $X(\mathbb{C})$, we consider the $G_{\mathbb{R}}$ -equivariant cohomology groups (resp. Tate cohomology)

$$\begin{aligned} H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) &:= H^i\left(R\Gamma(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n)))\right), \\ \widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) &:= H^i\left(R\widehat{\Gamma}(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), \mathbb{Z}(n)))\right). \end{aligned}$$

Of course, $X(\mathbb{C})$ is just a finite discrete space, so it is not necessary to use cohomology with compact support, but we use this notation for consistency with the general case considered in [2]. Since $\dim X(\mathbb{C}) = 0$, we have

$$\begin{aligned} H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) &\cong H^i(G_{\mathbb{R}}, H_c^0(X(\mathbb{C}), \mathbb{Z}(n))), \\ \widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) &\cong \widehat{H}^i(G_{\mathbb{R}}, H_c^0(X(\mathbb{C}), \mathbb{Z}(n))). \end{aligned}$$

PROPOSITION 4.1. *Let X be a one-dimensional arithmetic scheme with r_1 real places. Then the $G_{\mathbb{R}}$ -equivariant cohomology of $X(\mathbb{C})$ is*

$$\widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \equiv n \pmod{2}, \\ 0, & i \not\equiv n \pmod{2}; \end{cases} \quad (10)$$

$$H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} 0, & i < 0, \\ \mathbb{Z}^{\oplus d_n}, & i = 0, \\ \widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)), & i \geq 1. \end{cases} \quad (11)$$

Here d_n is the vanishing order given by (9).

Proof. We have

$$H_c^0(X(\mathbb{C}), \mathbb{Z}(n)) \cong \mathbb{Z}(n)^{\oplus r_1} \oplus (\mathbb{Z}(n) \oplus \mathbb{Z}(n))^{\oplus r_2},$$

and the $G_{\mathbb{R}}$ -action on the two summands is given by $x \mapsto \bar{x}$ and $(x, y) \mapsto (\bar{y}, \bar{x})$, respectively. (See Figure (1).)

We recall that the Tate cohomology of a finite cyclic group is 2-periodic:

$$\widehat{H}^i(G, A) \cong \begin{cases} \widehat{H}^0(G, A), & i \text{ even}, \\ \widehat{H}_0(G, A), & i \text{ odd}, \end{cases}$$

and the groups $\widehat{H}^0(G, A)$ and $\widehat{H}_0(G, A)$ are given by the exact sequence

$$0 \rightarrow \widehat{H}_0(G, A) \rightarrow A_G \xrightarrow{N} A^G \rightarrow \widehat{H}^0(G, A) \rightarrow 0$$

where N is the norm map induced by the action of $\sum_{g \in G} g$.

Therefore, we can consider two cases.

1) $G_{\mathbb{R}}$ -**module** $A = \mathbb{Z}(n)$ **with the action via** $x \mapsto \bar{x}$.

In this case, we see that

$$A^{G_{\mathbb{R}}} \cong \begin{cases} \mathbb{Z}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

Similarly, it is straightforward to calculate the coinvariants $A_{G_{\mathbb{R}}}$, and

$$\widehat{H}^0(G_{\mathbb{R}}, A) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n \text{ even,} \\ 0, & n \text{ odd,} \end{cases} \quad \widehat{H}_0(G_{\mathbb{R}}, A) \cong \begin{cases} 0, & n \text{ even,} \\ \mathbb{Z}/2\mathbb{Z}, & n \text{ odd.} \end{cases}$$

2) $G_{\mathbb{R}}$ -**module** $A = \mathbb{Z}(n) \oplus \mathbb{Z}(n)$ **with the action via** $(x, y) \mapsto (\bar{y}, \bar{x})$.

In this case $A^{G_{\mathbb{R}}} \cong \mathbb{Z}$ and $\widehat{H}^0(G_{\mathbb{R}}, A) = \widehat{H}_0(G_{\mathbb{R}}, A) = 0$.

Combining these two calculations, we obtain Tate cohomology groups (10). For the usual cohomology (11), we have

$$\begin{aligned} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) &\cong H_c^0(X(\mathbb{C}), \mathbb{Z}(n))^{G_{\mathbb{R}}}, \\ H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) &\cong \widehat{H}_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \quad \text{for } i \geq 1. \end{aligned} \quad \square$$

5 Étale motivic cohomology of one-dimensional schemes

In this section we review the structure of the étale motivic cohomology $H^i(X_{\text{ét}}, \mathbb{Z}^c(n))$ for one-dimensional X and $n < 0$. What follows is fairly well-known, so we claim no originality here, but we compile the references and state the result for a general one-dimensional arithmetic scheme.

PROPOSITION 5.1. *If X is a one-dimensional arithmetic scheme and $n < 0$, then*

$$H^i(X_{\text{ét}}, \mathbb{Z}^c(n)) \cong \begin{cases} 0, & i < -1, \\ \text{finitely generated of rk } d_n, & i = -1, \\ \text{finite,} & i = 0, 1, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \geq 2, i \not\equiv n \pmod{2}, \\ 0, & i \geq 2, i \equiv n \pmod{2}. \end{cases} \quad (12)$$

Here d_n is given by (9) and $r_1 = |X(\mathbb{R})|$ is the number of real places of X . Further, if $X = \text{Spec } \mathcal{O}_F$ for a number field F/\mathbb{Q} , then

$$H^1(X_{\text{ét}}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases} \quad (13)$$

An important ingredient of our proof is the arithmetic duality [2, Theorem I], which states that if $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated groups for all $i \in \mathbb{Z}$, then

$$\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \cong H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D, \quad (14)$$

where

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!} \mu_{p^r}^{\otimes n}[-1]. \quad (15)$$

Here $\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n))$ is the modified cohomology with compact support, for which we refer to [15, §2] and [2, Appendix B]. In particular,

$$\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \quad \text{if } X(\mathbb{R}) = \emptyset.$$

We recall that $(-)^D$ denotes the group $\text{Hom}(-, \mathbb{Q}/\mathbb{Z})$. We note that (14) is a powerful result, deduced in [2] from the work of Geisser [13].

Proof of Proposition 5.1. We use Lemma 2.1. We will say that $\mathcal{P}(X)$ holds if the motivic cohomology of X has the structure (12).

Let us first consider the case of a finite field $X = \text{Spec } \mathbb{F}_q$. We have

$$H^i(\text{Spec } \mathbb{F}_q, \mathbb{Z}^c(n)) \cong \begin{cases} \mathbb{Z}/(q^{-n} - 1), & i = 1, \\ 0, & i \neq 1. \end{cases} \quad (16)$$

—see, for example, [14, Example 4.2]. This is related to Quillen’s calculation of the K -theory of finite fields [35].

In general, if X is a zero-dimensional arithmetic scheme, then the motivic cohomology of X and X_{red} coincide, so we can assume that X is reduced. Then X is a finite disjoint union of $X_i = \text{Spec } \mathbb{F}_{q_i}$, and

$$H^i(X, \mathbb{Z}^c(n)) = \begin{cases} \text{finite}, & i = 1, \\ 0, & i \neq 1. \end{cases} \quad (17)$$

In particular, $\mathcal{P}(X)$ holds if $\dim X = 0$.

Now we check the compatibility properties for \mathcal{P} . If $X = \coprod_i X_i$ is a finite disjoint union, then $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \bigoplus_i H^i(X_{i, \acute{e}t}, \mathbb{Z}^c(n))$, hence the property \mathcal{P} is compatible with disjoint unions.

Similarly, if $U \subset X$ is a dense open subscheme, and $Z = X \setminus U$ its closed complement, then $\dim Z = 0$. We consider the long exact sequence (8). Since the cohomology of Z is concentrated in $i = 1$, we have $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong H^i(U_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i \neq 0, 1$, and what is left is an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^0(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow \\ H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow 0 \end{aligned}$$

Moreover, $d_n(X) = d_n(U)$. These considerations show that $\mathcal{P}(X)$ and $\mathcal{P}(U)$ are equivalent, and therefore Lemma 2.1 works, and it remains to establish $\mathcal{P}(X)$ for a curve X/\mathbb{F}_q or $X = \text{Spec } \mathcal{O}_F$.

Suppose that X/\mathbb{F}_q is a smooth curve. The groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated by [14, Proposition 4.3], so that the duality (14) holds. The \mathbb{Q}/\mathbb{Z} -dual groups

$$H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = \bigoplus_{\ell} H_c^{i-1}(X_{\acute{e}t}, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}(n))$$

are finite by [20, Theorem 3], and concentrated in $i = 1, 2, 3$ for dimension reasons. It follows that $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ in this case are finite groups concentrated in $i = -1, 0, 1$, and the property $\mathcal{P}(X)$ holds.

It remains to consider the case of $X = \text{Spec } \mathcal{O}_F$. In this case, the finite generation of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is also known; see, for example, [14, Proposition 4.14]. Therefore, the duality (14) holds. We have $\widehat{H}_c^i(\text{Spec } \mathcal{O}_F[1/p], \mu_{p^r}^{\otimes n}) = 0$ for $i \geq 3$ by Artin–Verdier duality [32, Chapter II, Corollary 3.3], or by [37, p. 268]. Therefore, it follows that $\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$ for $i \geq 4$, and hence by duality (14), $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for $i \leq -2$.

Now we identify the finite 2-torsion in $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i \geq 2$. By [8, Lemma 6.14], there is an exact sequence

$$\begin{aligned} \cdots \rightarrow H_c^{i-1}(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow \widehat{H}^{i-1}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow \\ \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow \cdots \end{aligned} \quad (18)$$

For $i \leq 0$ we have $H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$, and therefore

$$\widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \widehat{H}_c^{i-1}(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \not\equiv n \pmod{2}, \\ 0, & i \equiv n \pmod{2}. \end{cases}$$

By duality, for $i \geq 2$ we have

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & i \not\equiv n \pmod{2}, \\ 0, & i \equiv n \pmod{2}. \end{cases}$$

Now we determine the ranks of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i = -1, 0, 1$. By [26, Proposition 2.1] the Chern character for $i = -1, 0$

$$K_{-2n-i}(X) \rightarrow H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$$

has a finite 2-torsion kernel and cokernel. (Originally, the target group is defined over X_{Zar} , and we identify it with the cohomology on $X_{\acute{e}t}$ using the Beilinson–Lichtenbaum conjecture, which is now a theorem [11, Theorem 1.2]. We further use the isomorphism (7) to identify our motivic cohomology with the one used in [26].)

For $i = -1, 0$ we have therefore

$$\mathrm{rk}_{\mathbb{Z}} H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \mathrm{rk}_{\mathbb{Z}} K_{-2n-i}(X).$$

Together with Borel's calculation of the ranks of $K_m(\mathcal{O}_F)$ in [5], this implies that $H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is a finite group, while

$$\mathrm{rk}_{\mathbb{Z}} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) = d_n = \begin{cases} r_1 + r_2, & n \text{ even,} \\ r_2, & n \text{ odd.} \end{cases}$$

Finally, by [26, p.179] and (7), we have

$$H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

This concludes the proof. \square

6 Regulator for one-dimensional X

Now we explain what is meant by the regulator in our situation.

DEFINITION 6.1. We let the **regulator morphism** be the composition

$$\varrho_{X,n}: H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{x \mapsto x \otimes 1} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \otimes \mathbb{R} \xrightarrow{Reg_{X,n}} H_{BM}^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)),$$

where the map $Reg_{X,n}$ is defined in [3, §2].

The target is the Borel–Moore cohomology defined by

$$H_{BM}^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) := \mathrm{Hom}(H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}).$$

In general, the regulator takes values in Deligne–Beilinson cohomology, but the target simplifies in the case of $n < 0$, as explained in [3, §2].

REMARK 6.2. The only relevant group for the regulator is $H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))$, since the cohomology in other degrees is finite by Proposition 5.1.

The general definition in [3, §2] is based on the construction of Kerr, Lewis and Müller-Stach [23] which works at the level of complexes. This is not very important in the one-dimensional case, where the interesting cohomology is concentrated in $i = -1$. The reader can use any other equivalent construction of the regulator for motivic cohomology.

REMARK 6.3. If $X = \mathrm{Spec} \mathcal{O}_F$, then $\varrho_{X,n}$ can be identified with the Beilinson regulator map that appears in the special value conjecture of Flach and Morin in [8, §5.8.3].

LEMMA 6.4. For any one-dimensional arithmetic scheme X and $n < 0$, the \mathbb{R} -dual to the regulator

$$Reg_{X,n}^{\vee}: H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \rightarrow \text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})$$

is an isomorphism.

Proof. If X/\mathbb{F}_q , then the claim is trivial. For $X = \text{Spec } \mathcal{O}_F$, this is a well-known property of the Beilinson regulator. To apply Lemma 2.1, we need to check compatibility with disjoint unions and passing to a dense open subscheme $U \subset X$. For disjoint unions, this is clear. For a dense open subscheme $U \subset X$, the closed complement $Z = X \setminus U$ has dimension 0, and the localization exact sequence (8) with the long exact sequence for cohomology with compact support yields integral isomorphisms

$$\begin{aligned} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) &\xrightarrow{\cong} H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n)), \\ H_c^0(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{Z}(n)) &\xrightarrow{\cong} H_c^0(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{Z}(n)). \end{aligned}$$

We now have a commutative diagram

$$\begin{array}{ccc} H_c^0(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)) & \xrightarrow{Reg_{U,n}^{\vee}} & \text{Hom}(H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \\ \downarrow \cong & & \downarrow \cong \\ H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) & \xrightarrow{Reg_{X,n}^{\vee}} & \text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \end{array}$$

The upper arrow is an isomorphism if and only if the lower arrow is. \square

DEFINITION 6.5. For a one-dimensional arithmetic scheme X , we define the **regulator** to be

$$R_{X,n} := \text{vol}\left(\text{coker}\left(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\varrho_{X,n}} H_{BM}^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))\right)\right),$$

where the volume is taken with respect to the canonical integral structure.

If $X(\mathbb{C}) = \emptyset$, or n is odd and $r_2 = 0$, then $H_{BM}^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) = 0$, and we set $R_{X,n} = 1$.

LEMMA 6.6. Let X be a one-dimensional arithmetic scheme and $n < 0$. For any dense open subscheme $U \subset X$, we have $R_{X,n} = R_{U,n}$.

Proof. Follows from the proof of Lemma 6.4. \square

PROPOSITION 6.7. Given a one-dimensional arithmetic scheme X and $n < 0$, consider the two-term acyclic complex of real vector spaces

$$C^{\bullet}: 0 \rightarrow \underbrace{H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))}_{\text{deg } 0} \xrightarrow{Reg_{X,n}^{\vee}} \underbrace{\text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})}_{\text{deg } 1} \rightarrow 0$$

Then taking the determinant $\det_{\mathbb{R}}(C^\bullet)$ in the sense of Knudsen and Mumford [24], the image of the canonical map

$$\det_{\mathbb{Z}} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})^{-1} \rightarrow \\ \det_{\mathbb{R}} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \otimes_{\mathbb{R}} \det_{\mathbb{R}} \text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})^{-1} \xrightarrow{\cong} \mathbb{R}$$

corresponds to $R_{X,n} \mathbb{Z} \subset \mathbb{R}$.

Proof. In general, if F and F' are free groups of finite rank d , and

$$C^\bullet: 0 \rightarrow F \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\phi} F' \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow 0$$

is a two-term acyclic complex of real vector spaces, then the image of

$$\mathbb{Z} \cong \det_{\mathbb{Z}} F \otimes_{\mathbb{Z}} (\det_{\mathbb{Z}} F')^{-1} \rightarrow \det_{\mathbb{R}}(F \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \det_{\mathbb{R}}(F' \otimes_{\mathbb{Z}} \mathbb{R})^{-1} = \det_{\mathbb{R}}(C^\bullet) \xrightarrow{\cong} \mathbb{R}$$

corresponds to $D\mathbb{Z} \subset \mathbb{R}$, where D is the determinant of ϕ with respect to the bases induced by \mathbb{Z} -bases of F and F' . This follows from the explicit description of the isomorphism $\det_{\mathbb{R}}(C^\bullet) \xrightarrow{\cong} \mathbb{R}$ from [24, p. 33]: it is

$$\det_{\mathbb{R}}(F \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \det_{\mathbb{R}}(F' \otimes_{\mathbb{Z}} \mathbb{R})^{-1} \xrightarrow{\det_{\mathbb{R}}(\phi)} \det_{\mathbb{R}}(F' \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \det_{\mathbb{R}}(F \otimes_{\mathbb{Z}} \mathbb{R})^{-1} \xrightarrow{\cong} \mathbb{R}$$

where the last arrow is the canonical pairing.

Therefore, in our situation, the image of

$$\det_{\mathbb{Z}} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})^{-1}$$

is $D\mathbb{Z} \subset \mathbb{R}$, where D is the determinant of $\text{Reg}_{X,n}^{\vee}$ considered with respect to the bases induced by \mathbb{Z} -bases of $H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ and $\text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z})$. Dually, $D = R_{X,n}$. \square

7 Direct proof of the special value formula

In this section we explain how to prove our special value formula directly by combining the known special value formulas for $X = \text{Spec } \mathcal{O}_F$ and curves over finite fields X/\mathbb{F}_q via localization.

LEMMA 7.1. *Let $n < 0$.*

0) *If X is a zero-dimensional arithmetic scheme, then*

$$\zeta(X, n) = \pm \frac{1}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

1) *If X/\mathbb{F}_q is a curve over a finite field, then*

$$\zeta(X, n) = \pm \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

2) If $X = \text{Spec } \mathcal{O}_F$ for an abelian number field F/\mathbb{Q} , then

$$\zeta(X, n) = \pm \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))|} R_{X, n}.$$

In particular, formula (4) holds in these cases.

Proof. In part 0), motivic cohomology and the zeta function do not distinguish between X and X_{red} , so we can assume that X is a finite disjoint union of $\text{Spec } \mathbb{F}_{q_i}$. Thanks to (16),

$$\zeta(X, n) = \prod_i \frac{1}{1 - q_i^{-n}} = \pm \prod_i \frac{1}{|H^1(X_{i, \acute{e}t}, \mathbb{Z}^c(n))|} = \pm \frac{1}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

Note that this is formula (4) since $\delta = 0$ in this case and $H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) = H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ by (16).

For part 1), we refer the reader to [3, §5]. Part 2) follows from [8, Proposition 5.35]. The formula is equivalent to (4), since $2^\delta = |H^1(X_{\acute{e}t}, \mathbb{Z}(n))|$ by (13). \square

REMARK 7.2. The special value at $s = 0$ is not necessarily a rational number:

$$\zeta^*(\text{Spec } \mathbb{F}_q, 0) = \lim_{s \rightarrow 0} \frac{s}{1 - q^{-s}} = \frac{1}{\log q}.$$

Moreover,

$$H^i(\text{Spec } \mathbb{F}_{q, \acute{e}t}, \mathbb{Z}^c(0)) = \begin{cases} \mathbb{Z}, & i = 1, \\ \mathbb{Q}/\mathbb{Z}, & i = 2, \\ 0, & i \neq 1, 2. \end{cases}$$

This toy example already shows that it is important that we focus on the case of $n < 0$.

LEMMA 7.3. *Let X be a one-dimensional arithmetic scheme and let $U \subset X$ be a dense open subscheme. Then the special value formula (4) for X is equivalent to the corresponding formula for U .*

Proof. Let $Z = X \setminus U$ be the zero-dimensional complement. We have

$$\zeta(X, n) = \zeta(U, n) \zeta(Z, n),$$

where

$$\zeta(X, n) \stackrel{?}{=} \pm 2^\delta \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} R, \quad (19)$$

$$\zeta(U, n) \stackrel{?}{=} \pm 2^\delta \frac{|H^0(U_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(U_{\acute{e}t}, \mathbb{Z}^c(n))|} R, \quad (20)$$

$$\zeta(Z, n) = \pm \frac{1}{|H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

Here $\delta = \delta_{X,n} = \delta_{U,n}$, and $R = R_{X,n} = R_{U,n}$ (see Lemma 6.6). We note that $|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| = |H^{-1}(U_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}|$. On the other hand, the exact sequence of finite groups

$$0 \rightarrow H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^0(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow 0 \quad (21)$$

gives

$$\frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} = \frac{|H^0(U_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(U_{\acute{e}t}, \mathbb{Z}^c(n))|} \cdot \frac{1}{|H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n))|}.$$

From this we see that (19) and (20) are equivalent. \square

The above Lemmas 7.1 and 7.3 together with Lemma 2.1 now prove Theorem 1.2 from the introduction.

REMARK 7.4. Our proof of Lemma 7.1 uses [8, Proposition 5.35], which in turn reduces to the Tamagawa number conjecture for abelian F/\mathbb{Q} . The non-abelian version of Theorem 1.2 is therefore equivalent to the corresponding conjecture for non-abelian F/\mathbb{Q} .

REMARK 7.5. Note that $\zeta(\text{Spec } \mathbb{F}_q, n) = \frac{1}{1-q^{-n}} < 0$. Thus, if we remove m closed points from X , the sign of $\zeta^*(X, n)$ changes by $(-1)^m$. It is not hard to figure out the sign in any concrete example; however, it is not so clear in what terms to write the general expression for the sign.

8 Weil-étale cohomology of one-dimensional arithmetic schemes

In this section we calculate Weil-étale cohomology groups $H_{W,c}^i(X, \mathbb{Z}(n))$ for $n < 0$, as defined in [2]. Let us briefly recall the construction. In general, let X be an arithmetic scheme with finitely generated motivic cohomology $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$. The construction is carried out in two steps.

- **Step 1.** Consider the morphism in the derived category $\mathbf{D}(\mathbb{Z})$

$$\alpha_{X,n}: R\text{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \rightarrow R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

determined at the level of cohomology, using the arithmetic duality (14), by

$$H^i(\alpha_{X,n}): \text{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}} H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D \xleftarrow{\cong} \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)). \quad (22)$$

The complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined as a cone of $\alpha_{X,n}$:

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) &\xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\ &\rightarrow R\Gamma_{fg}(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{aligned}$$

The groups

$$H_{fg}^i(X, \mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X, \mathbb{Z}(n)))$$

are finitely generated for all $i \in \mathbb{Z}$, vanish for $i \ll 0$, and finite 2-torsion for $i \gg 0$. For the details we refer to [2, §5].

- **Step 2.** We consider a canonical morphism i_∞^* in the derived category $\mathbf{D}(\mathbb{Z})$ which is torsion and yields a commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ & \searrow^{u_\infty^*} & \swarrow_{i_\infty^*} \\ & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \end{array}$$

—see [2, §§6,7] for more details. Weil-étale cohomology with compact support is defined as a mapping fiber of i_∞^* :

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \rightarrow R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow [1]$$

The resulting groups

$$H_{W,c}^i(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n)))$$

are finitely generated and vanish for $i \notin [0, 2 \dim X + 1]$. We refer to [2, §7] for the general properties.

Here we calculate $H_{W,c}^i(X, \mathbb{Z}(n))$ for one-dimensional X .

PROPOSITION 8.1. *Let X be a one-dimensional arithmetic scheme and $n < 0$.*

0) $H_{W,c}^i(X, \mathbb{Z}(n)) = 0$ for $i \neq 1, 2, 3$.

1) *There is a short exact sequence*

$$0 \rightarrow \underbrace{H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))}_{\cong \mathbb{Z}^{\oplus d_n}} \rightarrow H_{W,c}^1(X, \mathbb{Z}(n)) \rightarrow T_1 \rightarrow 0 \quad (23)$$

in which T_1 sits in a short exact sequence of finite groups

$$0 \rightarrow \widehat{H}_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))^D \rightarrow T_1 \rightarrow 0$$

In particular, $H_{W,c}^1(X, \mathbb{Z}(n))$ is finitely generated of rank d_n , and

$$|T_1| = \frac{1}{2^\delta} \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|,$$

where δ is defined by (5).

2) There is an isomorphism of finitely generated groups

$$H_{W,c}^2(X, \mathbb{Z}(n)) \cong \underbrace{H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))^*}_{\cong \mathbb{Z}^{\oplus d_n}} \oplus \underbrace{H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D}_{\text{finite}}.$$

3) There is an isomorphism of finite groups

$$H_{W,c}^3(X, \mathbb{Z}(n)) \cong (H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{\text{tors}})^D.$$

We recall that $A^D := \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ and $A^* := \text{Hom}(A, \mathbb{Z})$.

Proof. From the definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$ we have a long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) &\xrightarrow{H^i(\alpha_{X,n})} H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \\ &\rightarrow H_{fg}^i(X, \mathbb{Z}(n)) \rightarrow \text{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \rightarrow \cdots \end{aligned} \quad (24)$$

Our calculations of motivic cohomology in Proposition 5.1 give

$$\text{Hom}(H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) = 0 \quad \text{for } i \neq -1,$$

and further by the definition of $\mathbb{Z}(n)$ in (15),

$$H_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) = 0 \quad \text{for } i \leq 0.$$

This implies that $H_{fg}^i(X, \mathbb{Z}(n)) = 0$ for $i \leq 0$. Since $H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) = 0$ for $i < 0$, we see from the exact sequence

$$\begin{aligned} \cdots \rightarrow H_{W,c}^i(X, \mathbb{Z}(n)) \rightarrow H_{fg}^i(X, \mathbb{Z}(n)) &\xrightarrow{H^i(i_{\infty}^*)} H_c^i(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \\ &\rightarrow H_{W,c}^{i+1}(X, \mathbb{Z}(n)) \rightarrow \cdots \end{aligned} \quad (25)$$

that $H_{W,c}^i(X, \mathbb{Z}(n)) = 0$ for $i \leq 0$.

For $i = 1$, the exact sequence (24) shows that $H_c^1(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow H_{fg}^1(X_{\acute{e}t}, \mathbb{Z}(n))$ is an isomorphism. Consequently, we see that $\ker H^1(i_{\infty}^*) \cong \ker H^1(u_{\infty}^*)$:

$$\begin{array}{ccc} H_c^1(X_{\acute{e}t}, \mathbb{Z}(n)) & \xrightarrow{\cong} & H_{fg}^1(X, \mathbb{Z}(n)) \\ H^1(u_{\infty}^*) \downarrow & & \swarrow H^1(i_{\infty}^*) \\ H_c^1(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & & \end{array}$$

From long exact sequences (25) and (18), we obtain short exact sequences

$$\begin{aligned} 0 \rightarrow H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow H_{W,c}^1(X, \mathbb{Z}(n)) \rightarrow \ker H^1(i_{\infty}^*) \rightarrow 0 \\ 0 \rightarrow \widehat{H}_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow \widehat{H}_c^1(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow \ker H^1(u_{\infty}^*) \rightarrow 0 \end{aligned}$$

Since $\ker H^1(i_\infty^*) \cong \ker H^1(u_\infty^*)$, this is part 1) of the proposition.

We proceed to compute $H_{W,c}^i(X, \mathbb{Z}(n))$ for $i \geq 2$. It is more convenient to do this without passing explicitly through $H_{fg}^i(X, \mathbb{Z}(n))$. Consider the morphism of complexes

$$\widehat{\alpha}_{X,n}: R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \rightarrow R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)),$$

defined in the same way as $\alpha_{X,n}$ in (22), only without the final projection from \widehat{H}_c^i to H_c^i :

$$H^i(\widehat{\alpha}_{X,n}): \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}} H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D \xleftarrow{\cong} \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)).$$

The relation between $\widehat{\alpha}_{X,n}$ and $\alpha_{X,n}$ is given by

$$\begin{array}{ccc} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\ & \searrow \alpha_{X,n} & \downarrow \\ & & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \end{array}$$

Here the vertical arrow comes from the definition of modified étale cohomology with compact support and it sits in an exact triangle

$$R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \xrightarrow{\widehat{u}_\infty^*} R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow \cdots [1]$$

—see [8, Lemma 6.14]. From the definition of $\widehat{\alpha}_{X,n}$ and the exact sequence (6), we calculate

$$\begin{aligned} \ker H^i(\widehat{\alpha}_{X,n}) &= H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^*, \\ \mathrm{coker} H^i(\widehat{\alpha}_{X,n}) &\cong (H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D. \end{aligned}$$

We denote a cone of $\widehat{\alpha}_{X,n}$ by $R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n))$ and set

$$\widehat{H}_{fg}^i(X, \mathbb{Z}(n)) := H^i(R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n))),$$

so that there is a long exact sequence

$$\cdots \rightarrow \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \xrightarrow{H^i(\widehat{\alpha}_{X,n})} \widehat{H}_c^i(X_{\acute{e}t}, \mathbb{Z}(n)) \rightarrow \widehat{H}_{fg}^i(X, \mathbb{Z}(n)) \rightarrow \mathrm{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \rightarrow \cdots$$

The corresponding short exact sequences

$$0 \rightarrow \mathrm{coker} H^i(\widehat{\alpha}_{X,n}) \rightarrow \widehat{H}_{fg}^i(X, \mathbb{Z}(n)) \rightarrow \ker H^{i+1}(\widehat{\alpha}_{X,n}) \rightarrow 0$$

are split, since $\ker H^{i+1}(\widehat{\alpha}_{X,n})$ is a free group. Therefore, we have

$$\widehat{H}_{fg}^i(X, \mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^* \oplus (H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D.$$

There is a commutative diagram with distinguished rows and columns

$$\begin{array}{ccccccc} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\widehat{\alpha}_{X,n}} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & [+1] \\ \downarrow id & & \downarrow & & \downarrow & & \downarrow id \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) & \xrightarrow{\alpha_{X,n}} & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & [+1] \\ \downarrow & & \downarrow \widehat{u}_\infty^* & & \downarrow \widehat{i}_\infty^* & & \downarrow \\ 0 & \longrightarrow & R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \xrightarrow{id} & R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) & \xrightarrow{\widehat{\alpha}_{X,n}[1]} & R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n))[1] & \longrightarrow & R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n))[1] & \longrightarrow & [+2] \end{array}$$

Here \widehat{u}_∞^* (resp. \widehat{i}_∞^*) is defined as the composition of the canonical morphism u_∞^* (resp. i_∞^*) with the projection to the Tate cohomology

$$\pi: R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \rightarrow R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)).$$

In our case of one-dimensional X , we know that $H^i(\pi)$ is an isomorphism for $i \geq 1$ (cf. [2, Proposition 3.2]). Therefore, the five-lemma applied to

$$\begin{array}{ccccccc} R\Gamma_{W,c}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \xrightarrow{i_\infty^*} & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & \cdots [1] \\ \downarrow f & & \downarrow id & & \downarrow \pi & & \downarrow f[1] \\ R\widehat{\Gamma}_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & R\Gamma_{fg}(X, \mathbb{Z}(n)) & \xrightarrow{\widehat{i}_\infty^*} & R\widehat{\Gamma}_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) & \longrightarrow & \cdots [1] \end{array}$$

shows that for $i \geq 2$ holds

$$H_{W,c}^i(X, \mathbb{Z}(n)) \cong \widehat{H}_{fg}^i(X, \mathbb{Z}(n)) \cong H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^* \oplus (H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D.$$

Our calculations of motivic cohomology in Proposition 5.1 yield

$$\begin{aligned} H_{W,c}^2(X, \mathbb{Z}(n)) &\cong H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))^* \oplus H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D, \\ H_{W,c}^3(X, \mathbb{Z}(n)) &\cong (H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D, \\ H_{W,c}^i(X, \mathbb{Z}(n)) &= 0 \text{ for } i \geq 4. \end{aligned} \quad \square$$

REMARK 8.2. A priori, the short exact sequence (23) need not split. This will not bother us for the determinant calculations in §9 below.

REMARK 8.3. The groups $H_{W,c}^i(X, \mathbb{Z}(n))$ for $X = \mathrm{Spec} \mathcal{O}_F$ are already calculated in [8, §5.8.3]. The result is (using the identification (7))

$$H_{W,c}^i(X, \mathbb{Z}(n)) \cong \begin{cases} \mathbb{Z}^{\oplus d_n}, & i = 1, \\ H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))^* \oplus H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D, & i = 2, \\ (H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D, & i = 3, \\ 0, & i \neq 1, 2, 3. \end{cases} \quad (26)$$

Our calculation generalizes this. What may look puzzling is the answer for $H_{W,c}^1(X, \mathbb{Z}(n))$ given by Proposition 8.1. In the case of $X = \text{Spec } \mathcal{O}_F$ we have, according to (13), that $H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r_1}$ for even n , and hence $T_1 = 0$, which agrees with (26).

Intuitively, the arithmetically interesting cohomology $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $X = \text{Spec } \mathcal{O}_F$ is concentrated in degrees $i = -1, 0$. The groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for $i \geq 1$ do not contain any interesting information: they are finite 2-torsion, coming from the real places of F . The transition to Weil-étale cohomology eliminates this 2-torsion. On the other hand, the group $H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))$ for a curve over a finite field X/\mathbb{F}_q is nontrivial and contains arithmetic information. The finite group T_1 appearing in the statement removes the 2-torsion coming from the real places of X .

REMARK 8.4. For a curve over a finite field X/\mathbb{F}_q , all groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finite, and our calculation gives $H_{W,c}^i(X, \mathbb{Z}(n)) \cong H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))^D$. This is true for any variety over a finite field X/\mathbb{F}_q and $n < 0$, under the assumption of finite generation of $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$; see [2, Proposition 7.7].

REMARK 8.5. It is conjectured in [3, §3] that

$$\text{ord}_{s=n} \zeta(X, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \text{rk}_{\mathbb{Z}} H_{W,c}^i(X, \mathbb{Z}(n)).$$

In this case

$$\begin{aligned} \text{rk}_{\mathbb{Z}} H_{W,c}^1(X, \mathbb{Z}(n)) &= \text{rk}_{\mathbb{Z}} H_{W,c}^2(X, \mathbb{Z}(n)) = d_n, \\ \text{rk}_{\mathbb{Z}} H_{W,c}^3(X, \mathbb{Z}(n)) &= 0, \end{aligned}$$

so the conjecture holds by Proposition 3.2.

9 Weil-étale proof of the special value formula

Now we explicitly write down the special value conjecture $\mathbf{C}(X, n)$ from [3, §4]. To do this, consider the canonical isomorphism

$$\begin{aligned} \lambda: \mathbb{R} &\xrightarrow{\cong} \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{R}} H_{W,c}^i(X, \mathbb{R}(n)))^{(-1)^i} \\ &\xrightarrow{\cong} \left(\bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{Z}} H_{W,c}^i(X, \mathbb{Z}(n)))^{(-1)^i} \right) \otimes_{\mathbb{Z}} \mathbb{R} \\ &\xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}, \end{aligned}$$

where the first isomorphism $\mathbb{R} \cong \bigotimes_{i \in \mathbb{Z}} (\det_{\mathbb{R}} H_{W,c}^i(X, \mathbb{R}(n)))^{(-1)^i}$ comes from the regulator, as explained below.

In our case, we are interested in the determinant of the Weil-étale complex

$$\begin{aligned} \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) &\cong \bigotimes_{i \in \mathbb{Z}} \det_{\mathbb{Z}} H_{W,c}^i(X, \mathbb{Z}(n))^{(-1)^i} \\ &= \det_{\mathbb{Z}} H_{W,c}^1(X, \mathbb{Z}(n))^{-1} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H_{W,c}^2(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H_{W,c}^3(X, \mathbb{Z}(n))^{-1}. \end{aligned}$$

Using the calculations from Proposition 8.1,

$$\begin{aligned} \det_{\mathbb{Z}} H_{W,c}^1(X, \mathbb{Z}(n)) &\cong \det_{\mathbb{Z}} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} T_1, \\ \det_{\mathbb{Z}} H_{W,c}^2(X, \mathbb{Z}(n)) &\cong \det_{\mathbb{Z}} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))^* \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D, \\ \det_{\mathbb{Z}} H_{W,c}^3(X, \mathbb{Z}(n)) &\cong \det_{\mathbb{Z}} (H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D. \end{aligned}$$

So we have an isomorphism (up to sign ± 1 , after rearranging the terms)

$$\begin{aligned} \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) &\cong \\ &= \det_{\mathbb{Z}} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))^{-1} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))^* \otimes_{\mathbb{Z}} \\ &\det_{\mathbb{Z}} (T_1)^{-1} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} ((H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D)^{-1}. \end{aligned}$$

Recall that T_1 , $H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D$, $(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D$ are finite groups, while the groups $H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ and $H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))^*$ are free of rank d_n . Now we consider the canonical trivialization

$$(\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \cong \bigotimes_{i \in \mathbb{Z}} \det_{\mathbb{R}} (H_{W,c}^i(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R}) \cong \mathbb{R}$$

via the regulator morphism

$$\begin{array}{ccc} H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n)) \otimes \mathbb{R} & & \text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}) \otimes \mathbb{R} \\ \downarrow \cong & & \downarrow \cong \\ H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) & \xrightarrow[\cong]{\text{Reg}_{X,n}^{\vee}} & \text{Hom}(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \end{array}$$

PROPOSITION 9.1. *Under the above trivialization, $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \subset \mathbb{R}$ corresponds to $\alpha^{-1} \mathbb{Z} \subset \mathbb{R}$, where*

$$\begin{aligned} \alpha &= \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D|}{|T_1| \cdot |(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D|} R_{X,n} \\ &= 2^{\delta} \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} R_{X,n}, \end{aligned}$$

the number δ is given by (5), and $R_{X,n}$ is the regulator from Definition 6.5.

Proof. For the finite groups T_1 , $H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))^D$, $(H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors})^D$, this is [3, Lemma A.5]. For the free groups $H_c^0(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))$ and $H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))^*$, on the other hand, this is Proposition 6.7 (now our groups sit in degrees 1 and 2, so the determinant gets inverted). \square

We recall that Conjecture $\mathbf{C}(X, n)$ from [3, §4] states that the canonical embedding $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \subset \mathbb{R}$ corresponds to $\zeta^*(X, n)^{-1} \mathbb{Z} \subset \mathbb{R}$.

PROPOSITION 9.2. *Let X be a one-dimensional arithmetic scheme and $n < 0$. Then the special value conjecture $\mathbf{C}(X, n)$ stated in [3] is equivalent to formula (4).*

In [3, §7] it is already proved (using essentially the same localization idea as in this text) that $\mathbf{C}(X, n)$ holds unconditionally for an abelian one-dimensional arithmetic scheme X . Together with the proposition above, this proves Theorem 1.2 from the introduction.

10 A couple of examples

We conclude with two examples that illustrate how localization arguments work. The first is rather general and consists in specifying §7 to the case of a non-maximal order in a number field.

EXAMPLE 10.1. Let $\mathcal{O} \subset \mathcal{O}_F$ be a non-maximal order in a number field F/\mathbb{Q} . Denote $X = \text{Spec } \mathcal{O}$ and $X' = \text{Spec } \mathcal{O}_F$. Geometrically, $\nu: X' \rightarrow X$ is the normalization. There exist open dense subschemes $U \subset X$ and $U' \subset X'$ such that ν induces an isomorphism $U' \cong U$. If we denote the corresponding closed complements by $Z = X \setminus U$ and $Z' = X' \setminus U'$, then we have

$$\zeta_{\mathcal{O}}(s) = \frac{\zeta(Z, s)}{\zeta(Z', s)} \zeta_F(s).$$

For this identity formulated in classical terms of algebraic number theory, see, for example, [18]. In particular,

$$\zeta_{\mathcal{O}}^*(n) = \pm \frac{|H^1(Z'_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n))|} \zeta_F^*(n).$$

Now our special value conjectures for $\zeta_{\mathcal{O}}^*(n)$ and $\zeta_F^*(n)$ take the form

$$\zeta_{\mathcal{O}}^*(n) \stackrel{?}{=} \pm 2^{\delta} \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} R, \quad (27)$$

$$\zeta_F^*(n) \stackrel{?}{=} \pm 2^{\delta} \frac{|H^0(X'_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^{-1}(X'_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| \cdot |H^1(X'_{\acute{e}t}, \mathbb{Z}^c(n))|} R. \quad (28)$$

Here $|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}| = |H^{-1}(X'_{\acute{e}t}, \mathbb{Z}^c(n))_{tors}|$, and the exact sequences of finite groups

$$\begin{aligned} 0 \rightarrow H^0(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^0(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow \\ H^1(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(X_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(U_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow 0 \end{aligned}$$

$$0 \rightarrow H^0(X'_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^0(U'_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(Z'_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(X'_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow H^1(U'_{\acute{e}t}, \mathbb{Z}^c(n)) \rightarrow 0$$

give us

$$\frac{|H^1(Z'_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} = \frac{|H^1(X'_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^1(X_{\acute{e}t}, \mathbb{Z}^c(n))|} \cdot \frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(n))|}{|H^0(X'_{\acute{e}t}, \mathbb{Z}^c(n))|},$$

which implies that the formulas (27) and (28) are equivalent.

The second example is suggested by [19, §7].

EXAMPLE 10.2. Let p be an odd prime. Consider the affine scheme

$$X = \mathrm{Spec}(\mathbb{Z}[1/2] \times_{\mathbb{F}_p} \mathbb{F}_p[t]) = \mathrm{Spec} \mathbb{Z}[1/2] \sqcup_{\mathrm{Spec} \mathbb{F}_p} \mathbb{A}_{\mathbb{F}_p}^1$$

obtained from $\mathrm{Spec} \mathbb{Z}[1/p]$ and $\mathbb{A}_{\mathbb{F}_p}^1 = \mathrm{Spec} \mathbb{F}_p[t]$ by gluing together the points corresponding to the prime ideals $(p) \subset \mathbb{Z}[1/2]$ and $(t) \subset \mathbb{F}_p[t]$:

$$\mathbb{Z}[1/2] \times_{\mathbb{F}_p} \mathbb{F}_p[t] = \{(a, f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv f(0) \pmod{p}\}.$$

If we take odd $n < 0$, then there is no regulator. Let us consider $n = -3$.

First, recall some calculations of the motivic cohomology of $\mathrm{Spec} \mathbb{Z}$. Using [26, Proposition 2.1] and known calculations of the K -groups of \mathbb{Z} (see Weibel's survey [38]), we get

$$\begin{aligned} H^{-1}(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong K_7(\mathbb{Z}) \cong \mathbb{Z}/240\mathbb{Z}, \\ H^0(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z}, \\ H^1(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) &= 0. \end{aligned}$$

We note that, as expected,

$$\zeta(\mathrm{Spec} \mathbb{Z}, -3) = \zeta(-3) = -\frac{B_4}{4} = \frac{1}{120} = \frac{|H^0(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3))|}{|H^{-1}(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3))|}.$$

The localization gives

$$\begin{aligned} H^{-1}(\mathrm{Spec} \mathbb{Z}[1/2]_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong H^{-1}(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/240\mathbb{Z}, \\ H^0(\mathrm{Spec} \mathbb{Z}[1/2]_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}, \\ H^1(\mathrm{Spec} \mathbb{Z}[1/2]_{\acute{e}t}, \mathbb{Z}^c(-3)) &= H^1(\mathrm{Spec} \mathbb{Z}_{\acute{e}t}, \mathbb{Z}^c(-3)) = 0. \end{aligned}$$

Arithmetically, this corresponds to the fact that the zeta function of $\mathrm{Spec} \mathbb{Z}[1/2]$ has the same Euler product as $\zeta(s)$, with the factor $\frac{1}{1-2^{-s}}$ removed. Therefore, when $s = -3$, the zeta-value should be corrected by $2^3 - 1 = 7$.

For $\mathbb{A}_{\mathbb{F}_p}^1$, we now have

$$H^i(\mathbb{A}_{\mathbb{F}_p}^1, \mathbb{Z}^c(n)) \cong H^{i+2}(\mathrm{Spec} \mathbb{F}_p, \mathbb{Z}^c(n-1)) \cong \begin{cases} \mathbb{Z}/(p^{1-n} - 1)\mathbb{Z}, & i = -1, \\ 0, & i \neq -1. \end{cases}$$

In particular, the motivic cohomology of $\mathbb{A}_{\mathbb{F}_p}^1$ is concentrated in

$$H^{-1}(\mathbb{A}_{\mathbb{F}_p, \acute{e}t}^1, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/(p^4 - 1)\mathbb{Z}.$$

Consider the normalization of X , given by $X' = \text{Spec } \mathbb{Z}[1/2] \sqcup \mathbb{A}_{\mathbb{F}_p}^1$:

$$\begin{array}{ccc} Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

Here $Z = \{\mathfrak{p}\}$, $Z' = \{\mathfrak{P}, \mathfrak{P}'\}$, and

$$\begin{aligned} \mathfrak{p} &:= \{(a, f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv f(0) \equiv 0 \pmod{p}\}, \\ \mathfrak{P} &:= \{(a, f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid a \equiv 0 \pmod{p}\}, \\ \mathfrak{P}' &:= \{(a, f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] \mid f(0) \equiv 0 \pmod{p}\}. \end{aligned}$$

The canonical morphism $X' \rightarrow X$ induces an isomorphism

$$X' \setminus Z' \cong X \setminus Z \cong (\text{Spec } \mathbb{Z} \setminus \{(2), (p)\}) \sqcup (\text{Spec } \mathbb{F}_p[t] \setminus (t)).$$

We calculate via localizations that

$$\begin{aligned} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong H^{-1}((X \setminus Z)_{\acute{e}t}, \mathbb{Z}^c(-3)) \cong \mathbb{Z}/240\mathbb{Z} \oplus \mathbb{Z}/(p^4 - 1)\mathbb{Z}, \\ H^0(X_{\acute{e}t}, \mathbb{Z}^c(-3)) &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/(p^3 - 1)\mathbb{Z}, \\ H^1(X_{\acute{e}t}, \mathbb{Z}^c(-3)) &= 0. \end{aligned}$$

Consequently,

$$\frac{|H^0(X_{\acute{e}t}, \mathbb{Z}^c(-3))|}{|H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(-3))| \cdot |H^1(X_{\acute{e}t}, \mathbb{Z}^c(-3))|} = \frac{7}{120} \frac{p^3 - 1}{p^4 - 1}.$$

At the level of zeta-functions,

$$\begin{aligned} \zeta(X, s) &= \zeta(Z, s) \zeta(X \setminus Z, s) = \frac{\zeta(Z, s)}{\zeta(Z', s)} \zeta(X', s) \\ &= \frac{1}{\zeta(\text{Spec } \mathbb{F}_p, s)} \zeta(\text{Spec } \mathbb{Z}[1/2], s) \zeta(\mathbb{A}_{\mathbb{F}_p}^1, s) \\ &= (1 - p^{-s})(1 - 2^{-s}) \zeta(s) \frac{1}{1 - p^{1-s}}. \end{aligned}$$

In particular, substituting $s = -3$, we get

$$\zeta(X, -3) = -\frac{7}{120} \frac{p^3 - 1}{p^4 - 1}.$$

References

- [1] A. A. Beilinson. Higher regulators and values of L -functions. In *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 181–238. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.
- [2] A. Beshenov. Weil-étale cohomology and duality for arithmetic schemes in negative weights, 2021. arXiv preprint 2012.11034.
- [3] A. Beshenov. Weil-étale cohomology and zeta-values of arithmetic schemes at negative integers, 2021. arXiv preprint 2102.12114.
- [4] S. Bloch. Algebraic cycles and higher K -theory. *Adv. in Math.*, 61(3):267–304, 1986.
- [5] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.
- [6] A. Borel. Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(4):613–636, 1977.
- [7] A. I. Borevich and I. R. Shafarevich. *Number theory*. Translated from the Russian by Newcomb Greenleaf. Pure and Applied Mathematics, Vol. 20. Academic Press, New York-London, 1966.
- [8] M. Flach and B. Morin. Weil-étale cohomology and zeta-values of proper regular arithmetic schemes. *Doc. Math.*, 23:1425–1560, 2018.
- [9] M. Flach and B. Morin. Compatibility of special value conjectures with the functional equation of zeta functions, 2020. preprint.
- [10] M. Flach and B. Morin. Deninger’s conjectures and Weil-Arakelov cohomology. *Münster J. Math.*, 13(2):519–540, 2020.
- [11] T. Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.
- [12] T. Geisser. Motivic cohomology, K -theory and topological cyclic homology. In *Handbook of K -theory. Vol. 1, 2*, pages 193–234. Springer, Berlin, 2005.
- [13] T. Geisser. Duality via cycle complexes. *Ann. of Math. (2)*, 172(2):1095–1126, 2010.
- [14] T. Geisser. On the structure of étale motivic cohomology. *J. Pure Appl. Algebra*, 221(7):1614–1628, 2017.
- [15] T. Geisser and A. Schmidt. Poitou-Tate duality for arithmetic schemes. *Compos. Math.*, 154(9):2020–2044, 2018.
- [16] A. B. Goncharov. Regulators. In *Handbook of K -theory. Vol. 1, 2*, pages 295–349. Springer, Berlin, 2005.

- [17] A. Huber and G. Kings. Bloch-Kato conjecture and Main Conjecture of Iwasawa theory for Dirichlet characters. *Duke Math. J.*, 119(3):393–464, 2003.
- [18] W. E. Jenner. On zeta functions of number fields. *Duke Math. J.*, 36:669–671, 1969.
- [19] B. W. Jordan and B. Poonen. The analytic class number formula for 1-dimensional affine schemes. *Bull. Lond. Math. Soc.*, 52(5):793–806, 2020.
- [20] B. Kahn. Some finiteness results for étale cohomology. *J. Number Theory*, 99(1):57–73, 2003.
- [21] B. Kahn. Algebraic K -theory, algebraic cycles and arithmetic geometry. In *Handbook of K -theory. Vol. 1, 2*, pages 351–428. Springer, Berlin, 2005.
- [22] N. M. Katz. Review of ℓ -adic cohomology. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 21–30. Amer. Math. Soc., Providence, RI, 1994.
- [23] M. Kerr, J. D. Lewis, and S. Müller-Stach. The Abel-Jacobi map for higher Chow groups. *Compos. Math.*, 142(2):374–396, 2006.
- [24] F. F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on “det” and “Div”. *Math. Scand.*, 39(1):19–55, 1976.
- [25] M. Kolster. K -theory and arithmetic. In *Contemporary developments in algebraic K -theory*, ICTP Lect. Notes, XV, pages 191–258. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.
- [26] M. Kolster and J. W. Sands. Annihilation of motivic cohomology groups in cyclic 2-extensions. *Ann. Sci. Math. Québec*, 32(2):175–187, 2008.
- [27] S. Lichtenbaum. Values of zeta-functions, étale cohomology, and algebraic K -theory. In *Algebraic K -theory, II: “Classical” algebraic K -theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 489–501. Lecture Notes in Math., Vol. 342, 1973.
- [28] S. Lichtenbaum. The Weil-étale topology on schemes over finite fields. *Compos. Math.*, 141(3):689–702, 2005.
- [29] S. Lichtenbaum. Euler characteristics and special values of zeta-functions. In *Motives and algebraic cycles*, volume 56 of *Fields Inst. Commun.*, pages 249–255. Amer. Math. Soc., Providence, RI, 2009.
- [30] S. Lichtenbaum. The Weil-étale topology for number rings. *Ann. of Math. (2)*, 170(2):657–683, 2009.
- [31] S. Lichtenbaum. Special values of zeta functions of schemes, 2021. preprint.

- [32] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.
- [33] B. Morin. Topological Hochschild homology and zeta-values, 2021. preprint.
- [34] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.
- [35] D. Quillen. On the cohomology and K -theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.
- [36] J.-P. Serre. Zeta and L functions. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue)*, pages 82–92. Harper & Row, New York, 1965.
- [37] C. Soulé. K -théorie des anneaux d’entiers de corps de nombres et cohomologie étale. *Invent. Math.*, 55(3):251–295, 1979.
- [38] C. Weibel. Algebraic K -theory of rings of integers in local and global fields. In *Handbook of K -theory. Vol. 1, 2*, pages 139–190. Springer, Berlin, 2005.

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