Weil-étale cohomology and zeta-values of arithmetic schemes at negative integers

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Abstract

Following the ideas of Flach and Morin [11], we state a conjecture in terms of Weil-étale cohomology for the vanishing order and special value of the zeta function $\zeta(X,s)$ at $s = n < 0$, where X is a separated scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. We prove that the conjecture is compatible with closed-open decompositions of schemes and with affine bundles, and consequently, that it holds for cellular schemes over certain one-dimensional bases.

This is a continuation of [2], which gives a construction of Weil-étale cohomology for $n < 0$ under the mentioned assumptions on X.

1 Introduction

Let X be an **arithmetic scheme**, by which we mean in this paper that it is a separated scheme of finite type $X \to \operatorname{Spec}\mathbb{Z}$. Then the corresponding **zeta function** is defined by

$$\zeta(X,s) = \prod_{\substack{x \in X \\ \text{closed pt.}}} \frac{1}{1 - N(x)^{-s}}. \tag{1.1}$$

Here, for a closed point $x \in X$, the norm

$$N(x) = |\kappa(x)| = |\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}|$$

is the size of the corresponding residue field. The product converges for $\operatorname{Re} s > \dim X$, and conjecturally admits a meromorphic continuation to the whole complex plane. Basic facts and conjectures about zeta functions of schemes can be found in [42].

Of particular interest are the so-called special values of $\zeta(X,s)$ at integers $s = n \in \mathbb{Z}$, also known as the **zeta-values** of X. To define these, we assume that $\zeta(X,s)$ admits a meromorphic continuation around $s = n$. We denote by

$$d_n = \operatorname{ord}_{s=n} \zeta(X,s)$$

the **vanishing order** of $\zeta(X,s)$ at $s = n$. That is, $d_n > 0$ (resp. $d_n < 0$) if $\zeta(X,s)$ has a zero (resp. pole) of order $|d_n|$ at $s = n$.

The **special value** of $\zeta(X,s)$ at $s = n$ is defined as the leading nonzero coefficient of the Taylor expansion:

$$\zeta^*(X,n) = \lim_{s\to n} (s-n)^{-d_n}\, \zeta(X,s).$$

Early on, Lichtenbaum conjectured that both numbers $\operatorname{ord}_{s=n}\zeta(X,s)$ and $\zeta^*(X,n)$ should have a cohomological interpretation related to the étale motivic cohomology of X (see e.g. [33] for varieties over finite fields).

This is made precise in Lichtenbaum's Weil-étale program. It suggests the existence of **Weil-étale cohomology**, which is a suitable modification of motivic cohomology that encodes the information about the vanishing order and the special value of $\zeta(X,s)$ at $s=n$. For Lichtenbaum's recent work on this topic, we refer the reader to [34, 35, 36, 37].

The case of varieties over finite fields $X/\mathbb{F}_q$ is now well understood thanks to the work of Geisser [15, 16, 17].

Flach and Morin considered the case of proper, regular arithmetic schemes X. In [10] they have studied the corresponding Weil-étale topos. Later, in [39] Morin gave an explicit construction of Weil-étale cohomology groups $H^i_{W,c}(X,\mathbb{Z})$ for a proper and regular arithmetic scheme X. This construction was further generalized by Flach and Morin in [11] to groups $H^i_{W,c}(X,\mathbb{Z}(n))$ with weights $n\in\mathbb{Z}$, again for a proper and regular X.

Motivated by the work of Flach and Morin, the author constructed in [2] Weil-étale cohomology groups $H^i_{W,c}(X,\mathbb{Z}(n))$ for any arithmetic scheme X (removing the assumption that X is proper or regular) and strictly negative weights $n<0$. The construction is based on the following assumption.

Conjecture. $\mathbf{L}^c(X_{\acute{e}t},n)$: given an arithmetic scheme X and $n<0$, the cohomology groups $H^i(X_{\acute{e}t},\mathbb{Z}^c(n))$ are finitely generated for all $i\in\mathbb{Z}$.

For the known cases, see [2, §8]. Under this conjecture, we constructed in [2, §7] perfect complexes of abelian groups $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ and the corresponding cohomology groups

$$H^i_{W,c}(X,\mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X,\mathbb{Z}(n))).$$

This text is a continuation of [2] and investigates the conjectural relation of our Weil-étale cohomology to the special value of $\zeta(X,s)$ at $s=n<0$. Specifically, we make the following conjectures.

1) **Conjecture VO**(X,n) (see §3): *the vanishing order is given by the weighted alternating sum of ranks*

$$\operatorname{ord}_{s=n}\zeta(X,s) = \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

2) A consequence of **Conjecture B**(X,n) (see §2 and Lemma 4.2): *after tensoring the cohomology groups $H^i_{W,c}(X,\mathbb{Z}(n))$ with $\mathbb{R}$, we obtain a long exact sequence of finite dimensional real vector spaces*

$$\cdots\to H^{i-1}_{W,c}(X,\mathbb{R}(n)) \xrightarrow{\smile\theta} H^i_{W,c}(X,\mathbb{R}(n)) \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X,\mathbb{R}(n))\to\cdots$$

It follows that there is a canonical isomorphism

$$\lambda\colon \mathbb{R}\xrightarrow{\cong}(\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)))\otimes\mathbb{R}.$$

Here $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ is the determinant of the perfect complex of abelian groups $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, in the sense of Knudsen and Mumford

[31]. In particular, $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is a free $\mathbb{Z}$-module of rank 1. For the convenience of the reader, we give a brief overview of determinants in Appendix A.

3) **Conjecture** $\mathbf{C}(X, n)$ (see §4): *the special value is determined up to sign by*
$$\lambda(\zeta^*(X, n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)).$$

If X is proper and regular, then our construction of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ and the above conjectures agree with those of Flach and Morin from [11]. Apart from removing the assumption that X is proper and regular, a novelty of this work is that we prove the compatibility of the conjectures with operations on schemes, in particular with closed-open decompositions $Z \not\hookrightarrow X \hookleftarrow U$, where $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is the open complement, and with affine bundles $\mathbb{A}^r_X = \mathbb{A}^r_{\mathbb{Z}} \times X$ (see Proposition 6.3 and Theorem 6.8). This gives a machinery for starting from the cases of schemes for which the conjectures are known and constructing new schemes for which the conjectures also hold. As an application, we prove in §7 the following result.

MAIN THEOREM. *Let B be a one-dimensional arithmetic scheme, such that each of the generic points $\eta \in B$ satisfies one of the following properties:*

a) $\operatorname{char} \kappa(\eta) = p > 0$*;*

b) $\operatorname{char} \kappa(\eta) = 0$*, and* $\kappa(\eta)/\mathbb{Q}$ *is an abelian number field.*

If X is a B-cellular arithmetic scheme with smooth quasi-projective fiber $X_{red,\mathbb{C}}$, then Conjectures $\mathbf{VO}(X, n)$ *and* $\mathbf{C}(X, n)$ *hold unconditionally for any $n < 0$.*

In fact, this result is established for a larger class of arithmetic schemes $\mathcal{C}(\mathbb{Z})$; we refer to §7 for more details.

Outline of the paper

In §2 we define the regulator morphism, based on the construction of Kerr, Lewis, and Müller-Stach [30], and state the associated Conjecture $\mathbf{B}(X, n)$.

Then §3 is devoted to Conjecture $\mathbf{VO}(X, n)$ about the vanishing order. We also explain why it is consistent with a conjecture of Soulé, and with the vanishing order arising from the expected functional equation.

In §4 we state Conjecture $\mathbf{C}(X, n)$ about the special value.

We explain in §5 that if X is a variety over a finite field, then Conjecture $\mathbf{C}(X, n)$ is consistent with the conjectures considered by Geisser in [15, 16, 17], and it follows from Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$.

Then we prove in §6 that Conjectures $\mathbf{VO}(X, n)$ and $\mathbf{C}(X, n)$ are compatible with basic operations on schemes: disjoint unions, closed-open decompositions, and affine bundles. Using these results, we conclude in §7 with a class of schemes for which the conjectures hold unconditionally.

For the convenience of the reader, Appendix A gives a brief overview of basic definitions and facts related to the determinants of complexes.

Notation

In this paper, X always denotes an **arithmetic scheme** (separated, of finite type over $\operatorname{Spec}\mathbb{Z}$), and n is always a strictly negative integer.

We denote by

$$R\Gamma_{fg}(X,\mathbb{Z}(n)) \quad \text{and} \quad R\Gamma_{W,c}(X,\mathbb{Z}(n))$$

the complexes of abelian groups constructed in [2] under Conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$. We set

$$H^i_{fg}(X,\mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X,\mathbb{Z}(n))),$$
$$H^i_{W,c}(X,\mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X,\mathbb{Z}(n))).$$

By [2, Proposition 5.5 and 7.12], these cohomology groups are finitely generated, assuming Conjecture $\mathbf{L}^c(X_{\acute{e}t},n)$; moreover, the groups $H^i_{W,c}(X,\mathbb{Z}(n))$ are bounded, and $H^i_{fg}(X,\mathbb{Z}(n))$ are bounded from below and finite 2-torsion for $i \gg 0$.

Briefly, the construction fits in the following diagram of distinguished triangles in the derived category $\mathbf{D}(\mathbb{Z})$:

$$\begin{tikzcd}
 & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r] \arrow[d, "\alpha_{X,n}"] & 0 \arrow[d] & \\
 & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, "u_\infty^*"] \arrow[d] & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[d, "id"] & \\
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, dashed, "i_\infty^*"] \arrow[d] & R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & {[1]} \\
 & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \arrow[r] & 0 &
\end{tikzcd}$$

For more details, see [2]. For real coefficients, we set

$$R\Gamma_{fg}(X,\mathbb{R}(n)) := R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{R},$$
$$R\Gamma_{W,c}(X,\mathbb{R}(n)) := R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R}.$$

Accordingly,

$$H^i_{fg}(X,\mathbb{R}(n)) := H^i(R\Gamma_{fg}(X,\mathbb{R}(n))) = H^i_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{R},$$
$$H^i_{W,c}(X,\mathbb{R}(n)) := H^i(R\Gamma_{W,c}(X,\mathbb{R}(n))) = H^i_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R}.$$

By $X(\mathbb{C})$ we denote the space of complex points of X with the usual analytic topology. It carries a natural action of $G_\mathbb{R} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ via the complex conjugation. For a subring $A \subseteq \mathbb{R}$ we denote by $A(n)$ the $G_\mathbb{R}$-module $(2\pi i)^n\, A$, and also the corresponding constant $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$.

We denote by $R\Gamma_c(X(\mathbb{C}),A(n))$ the cohomology with compact support with $A(n)$-coefficients, and its $G_\mathbb{R}$-equivariant version is defined by

$$R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),A(n)) := R\Gamma(G_\mathbb{R},R\Gamma_c(X(\mathbb{C}),A(n)))$$

For real coefficients, we have

$$H^i_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) = H^i_c(X(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}},$$

where the $G_\mathbb{R}$-action on $H^i_c(X(\mathbb{C}), \mathbb{R}(n))$ naturally comes from the corresponding action on $X(\mathbb{C})$ and $\mathbb{R}(n)$.

Borel–Moore homology is defined as the dual to cohomology with compact support. We are interested in the real coefficients:

$$\begin{aligned} R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n)) &:= R\mathrm{Hom}(R\Gamma_c(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}), \\ R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) &:= R\mathrm{Hom}(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R}). \end{aligned}$$

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2 Regulator morphism and Conjecture B(X, n)

In order to formulate the special value conjecture, we need a regulator morphism from motivic cohomology to Deligne(–Beilinson) (co)homology. Such regulators were originally introduced by Bloch in [5], and here we use the construction of Kerr, Lewis, and Müller-Stach [30], which works at the level of complexes. We will simply call it "the KLM regulator." It works under the assumption that $X_{red,\mathbb{C}}$ is a smooth quasi-projective variety.

For simplicity, in this section we assume that X is reduced (motivic cohomology does not distinguish between X and X_{red}), and that $X_\mathbb{C}$ is connected of dimension $d_\mathbb{C}$ (otherwise, the arguments below can be applied to each connected component). We fix a compactification by a normal crossing divisor

$$\begin{tikzcd} X_\mathbb{C} \arrow[r, hook, "j"] & \overline{X}_\mathbb{C} & D \arrow[l, hook'] \end{tikzcd}$$

The KLM regulator has the form of a morphism in the derived category

$$z^p(X_\mathbb{C}, -\bullet) \otimes \mathbb{Q} \to {}'C^{2p-2d_\mathbb{C}+\bullet}_\mathcal{D}(\overline{X}_\mathbb{C}, D, \mathbb{Q}(p - d_\mathbb{C})). \tag{2.1}$$

Here $z^p(X_\mathbb{C}, -\bullet)$ denotes the Bloch's cycle complex [4]. To define it, consider the algebraic simplex $\Delta^i_\mathbb{C} = \operatorname{Spec} \mathbb{C}[t_0, \ldots, t_i]/(1 - \sum_j t_j)$. Then, $z^p(X_\mathbb{C}, i)$ is freely generated by algebraic cycles $Z \subset X_\mathbb{C} \times_{\operatorname{Spec} \mathbb{C}} \Delta^i_\mathbb{C}$ of codimension p which intersect the faces properly. It is more convenient for us to work with

$$z_{d_\mathbb{C}-p}(X_\mathbb{C}, i) = z^p(X_\mathbb{C}, i),$$

generated by cycles $Z \subset X_{\mathbb{C}} \times_{\operatorname{Spec}\mathbb{C}} \Delta^i_{\mathbb{C}}$ of dimension $p+i$.

The complex ${}'C^\bullet_{\mathcal{D}}(\overline{X}_{\mathbb{C}}, D, \mathbb{Q}(k))$ on the right-hand side of (2.1) computes Deligne(–Beilinson) homology, as defined by Jannsen [22]. If we take $p = d_{\mathbb{C}} + 1 - n$, tensor it with $\mathbb{R}$ and shift it by $2n$, we obtain

$$z_{n-1}(X_{\mathbb{C}}, -\bullet) \otimes \mathbb{R}[2n] \to {}'C^{2+\bullet}_{\mathcal{D}}(\overline{X}_{\mathbb{C}}, D, \mathbb{R}(1-n)). \tag{2.2}$$

Remark 2.1. Some comments are in order.

1. Originally, the KLM regulator is defined using a cubical version of cycle complexes, but these are quasi-isomorphic to the usual simplicial cycle complexes by [32], so we make no distinction here. For an explicit simplicial version of the KLM regulator, see [29].

2. The KLM regulator is defined as a true morphism of complexes (not just a morphism in the derived category) on a subcomplex $z^r_{\mathbb{R}}(X_{\mathbb{C}}, \bullet) \subset z^r(X_{\mathbb{C}}, \bullet)$. This inclusion becomes a quasi-isomorphism if we pass to rational coefficients. In the original paper [30] this is stated without tensoring with $\mathbb{Q}$, but the omission is acknowledged later in [28]. For our purposes, it suffices to have a regulator with coefficients in $\mathbb{R}$.

3. The case of a smooth quasi-projective $X_{\mathbb{C}}$, where one must consider a compactification by a normal crossing divisor as above, is treated in [30, §5.9].

Now we make a small digression to identify the right-hand side of (2.2). Under our assumption that $n < 0$, Deligne homology is equivalent to Borel–Moore homology.

Lemma 2.2. *For any $n < 0$ there is a quasi-isomorphism*

$$\begin{aligned} {}'C^\bullet_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, D, \mathbb{R}(1-n)) &\cong R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n))[-1] \\ &:= R\mathrm{Hom}(R\Gamma_c(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R})[-1]. \end{aligned}$$

Moreover, it respects the natural actions of $G_{\mathbb{R}}$ on both complexes.

Proof. From the proof of [22, Theorem 1.15], for any $k \in \mathbb{Z}$ we have a quasi-isomorphism

$${}'C^\bullet_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, D, \mathbb{R}(k)) \cong R\Gamma(\overline{X}(\mathbb{C}), \mathbb{R}(k+d_{\mathbb{C}})_{\mathcal{D}\text{-}\mathcal{B},(\overline{X}_{\mathbb{C}}, X_{\mathbb{C}})})[2d_{\mathbb{C}}], \tag{2.3}$$

where

$$\mathbb{R}(k+d_{\mathbb{C}})_{\mathcal{D}\text{-}\mathcal{B},(\overline{X}_{\mathbb{C}}, X_{\mathbb{C}})} = \operatorname{Cone}\left(\begin{array}{c} Rj_*\mathbb{R}(k+d_{\mathbb{C}}) \\ \oplus \\ \Omega^{\geqslant k+d_{\mathbb{C}}}_{\overline{X}(\mathbb{C})}(\log D) \end{array} \xrightarrow{\epsilon-\iota} Rj_*\Omega^\bullet_{X(\mathbb{C})} \right)[-1]$$

is the sheaf whose hypercohomology on $\overline{X}(\mathbb{C})$ gives Deligne–Beilinson cohomology (see [9] for more details).

Here $\Omega^\bullet_{\overline{X}(\mathbb{C})}$ denotes the usual de Rham complex of holomorphic differential forms, and $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D)$ is the complex of forms with at most logarithmic poles along $D(\mathbb{C})$. The latter complex is filtered by subcomplexes

$\Omega^{\geqslant\bullet}_{\overline{X}(\mathbb{C})}(\log D)$. The morphism $\epsilon\colon Rj_*\mathbb{R}(k) \to Rj_*\Omega^\bullet_{X(\mathbb{C})}$ is induced by the canonical morphism of sheaves $\mathbb{R}(k) \to \mathcal{O}_{X(\mathbb{C})}$, and ι is induced by the natural inclusion $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D) \xrightarrow{\cong} j_*\Omega^\bullet_{X(\mathbb{C})} = Rj_*\Omega^\bullet_{X(\mathbb{C})}$, which is a quasi-isomorphism of filtered complexes.

We are interested in the case of $k > 0$ when the part $\Omega^{\geqslant k+d_\mathbb{C}}_{\overline{X}(\mathbb{C})}(\log D)$ vanishes, and we obtain

$$\begin{aligned}
\mathbb{R}(k+d_\mathbb{C})_{\mathcal{D}\text{-}\mathcal{B},(\overline{X}_\mathbb{C},X_\mathbb{C})} &\cong Rj_*\operatorname{Cone}\Big(\mathbb{R}(k+d_\mathbb{C}) \xrightarrow{\epsilon} \Omega^\bullet_{X(\mathbb{C})}\Big)[-1] \\
&\cong Rj_*\Big(\mathbb{R}(k+d_\mathbb{C}) \xrightarrow{\epsilon} \Omega^\bullet_{X(\mathbb{C})}[-1]\Big) \\
&\cong Rj_*\Big(\mathbb{R}(k+d_\mathbb{C}) \to \mathbb{C}[-1]\Big) \qquad (2.4) \\
&\cong Rj_*\mathbb{R}(k+d_\mathbb{C}-1)[-1] \qquad (2.5)
\end{aligned}$$

Here (2.4) comes from the Poincaré lemma $\mathbb{C} \cong \Omega^\bullet_{X(\mathbb{C})}$ and (2.5) from the short exact sequence of $G_\mathbb{R}$-modules $\mathbb{R}(k+d_\mathbb{C}) \rightarrowtail \mathbb{C} \twoheadrightarrow \mathbb{R}(k+d_\mathbb{C}-1)$.

Returning to (2.3) for $k = 1-n$, we find that

$$\begin{aligned}
{}'C^\bullet_\mathcal{D}(\overline{X_\mathbb{C}}, D, \mathbb{R}(1-n)) &\cong R\Gamma(X(\mathbb{C}), \mathbb{R}(d_\mathbb{C}-n))[2d_\mathbb{C}-1] \\
&\cong R\mathrm{Hom}(R\Gamma_c(X(\mathbb{C}),\mathbb{R}(n)),\mathbb{R})[-1].
\end{aligned}$$

Here the final isomorphism is Poincaré duality. All the above is $G_\mathbb{R}$-equivariant. $\square$

Returning now to (2.2), the previous lemma allows us to reinterpret the KLM regulator as

$$z_{n-1}(X_\mathbb{C}, -\bullet) \otimes \mathbb{R}[2n] \to R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R})[1]. \qquad (2.6)$$

We have

$$\begin{aligned}
z_{n-1}(X_\mathbb{C}, -\bullet) \otimes \mathbb{R}[2n] &= z_{n-1}(X_\mathbb{C}, -\bullet) \otimes \mathbb{R}[2n-2][2] \\
&= \Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2], \qquad (2.7)
\end{aligned}$$

where the complex of sheaves $\mathbb{R}^c(p)$ is defined by $U \rightsquigarrow z_p(U, -\bullet) \otimes \mathbb{R}[2p]$. By étale cohomological descent [18, Theorem 3.1],

$$\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1)) \cong R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1)). \qquad (2.8)$$

(We note that [18, Theorem 3.1] holds unconditionally, since the Beilinson–Lichtenbaum conjecture follows from the Bloch–Kato conjecture, which is now a theorem; see also [14] where the consequences of Bloch–Kato for motivic cohomology are deduced.)

Finally, the base change from X to $X_\mathbb{C}$ naturally maps cycles $Z \subset X \times \Delta^i_\mathbb{Z}$ of dimension n to cycles in $X_\mathbb{C} \times_{\operatorname{Spec}\mathbb{C}} \Delta^i_\mathbb{C}$ of dimension $n-1$, so that there is a morphism

$$R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2]. \qquad (2.9)$$

Remark 2.3. Assuming that X is flat and has pure Krull dimension d, we have $\mathbb{R}^c(n)^X = \mathbb{R}(d-n)^X[2d]$, where $\mathbb{R}(\bullet)$ is the usual cycle complex defined by

$z^n(\textvisiblespace, -\bullet)[-2n]$. Similarly, $\mathbb{R}^c(n)^{X_\mathbb{C}} = \mathbb{R}(d_\mathbb{C} - n)^{X_\mathbb{C}}[2d_\mathbb{C}]$, with $d_\mathbb{C} = d - 1$. With this renumbering, the morphism (2.9) becomes

$$R\Gamma(X_{\acute{e}t}, \mathbb{R}(d-n))[2d] \to R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}(d-n))[2d].$$

This probably looks more natural, but we make no additional assumptions about X and work exclusively with complexes $A^c(\bullet)$ defined in terms of dimension of algebraic cycles, rather than $A(\bullet)$ defined in terms of codimension.

Definition 2.4. Given an arithmetic scheme X with smooth quasi-projective $X_\mathbb{C}$ and $n < 0$, consider the composition of morphisms

$$\begin{aligned} R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \xrightarrow{(2.9)} R\Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2] \overset{(2.8)}{\cong} \Gamma(X_{\mathbb{C},\acute{e}t}, \mathbb{R}^c(n-1))[2] \\ \overset{(2.7)}{=} z_{n-1}(X_\mathbb{C}, -\bullet)_\mathbb{R}[2n] \xrightarrow{(2.6)} R\Gamma_{BM}(X(\mathbb{C}), \mathbb{R}(n)), \mathbb{R})[1]. \end{aligned}$$

Moreover, we take the $G_\mathbb{R}$-invariants, which gives us the **(étale) regulator**

$$Reg_{X,n} \colon R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[1].$$

Now we state our conjecture about the regulator, which will play an important role in everything that follows.

Conjecture 2.5. $\mathbf{B}(X,n)$: given an arithmetic scheme X with smooth quasi-projective $X_\mathbb{C}$ and $n < 0$, the regulator morphism $Reg_{X,n}$ induces a quasi-isomorphism of complexes of real vector spaces

$$Reg^\vee_{X,n} \colon R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1] \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}).$$

Remark 2.6. If $X/\mathbb{F}_q$ is a variety over a finite field, then $X(\mathbb{C}) = \varnothing$, so the regulator map is not interesting. Indeed, in our setting, its purpose is to take care of the Archimedian places of X. In this case $\mathbf{B}(X,n)$ implies that $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are torsion groups. However, by [2, Proposition 4.2], Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ already implies that $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finite groups.

Remark 2.7. We reiterate that our construction of $Reg_{X,n}$ works for $X_{red,\mathbb{C}}$ smooth quasi-projective. In everything that follows, whenever the regulator morphism or Conjecture $\mathbf{B}(X,n)$ is brought, we tacitly assume this restriction. This is rather unfortunate, since Weil-étale cohomology was constructed in [2] for any arithmetic scheme, assuming only Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$. Defining the regulator for singular $X_{red,\mathbb{C}}$ is an interesting project for future work.

3 Vanishing order Conjecture $\mathbf{VO}(X,n)$

Assuming that $\zeta(X,s)$ admits a meromorphic continuation around $s = n < 0$, we make the following conjecture for the vanishing order at $s = n$.

Conjecture 3.1. $\mathbf{VO}(X,n)$: one has

$$\mathrm{ord}_{s=n}\, \zeta(X,s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \mathrm{rk}_\mathbb{Z}\, H^i_{W,c}(X, \mathbb{Z}(n)).$$

We note that the right-hand side makes sense under Conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, which implies that $H^i_{W,c}(X, \mathbb{Z}(n))$ are finitely generated groups, trivial for $|i| \gg 0$; see [2, Proposition 7.12].

Remark 3.2. Conjecture $\mathbf{VO}(X, n)$ is similar to [11, Conjecture 5.11]. If X is proper and regular, then $\mathbf{VO}(X, n)$ is the same as Flach and Morin's vanishing order conjecture. Indeed, the latter is

$$\operatorname{ord}_{s=n} \zeta(X, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_{ar,c}(X, \widetilde{\mathbb{R}}(n)), \tag{3.1}$$

where

$$R\Gamma_{ar,c}(X, \widetilde{\mathbb{R}}(n)) := R\Gamma_c(X, \mathbb{R}(n)) \oplus R\Gamma_c(X, \mathbb{R}(n))[-1].$$

Moreover, [11, Proposition 4.14], gives a distinguished triangle

$$\begin{aligned} R\Gamma_{dR}(X_{\mathbb{R}}/\mathbb{R})/\operatorname{Fil}^n[-2] \to R\Gamma_{ar,c}(X, \widetilde{\mathbb{R}}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \\ \to R\Gamma_{dR}(X_{\mathbb{R}}/\mathbb{R})/\operatorname{Fil}^n[-1] \end{aligned}$$

So, in case of $n < 0$ we have $R\Gamma_{ar,c}(X, \widetilde{\mathbb{R}}(n)) \cong R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$ and (3.1) is exactly Conjecture $\mathbf{VO}(X, n)$.

Remark 3.3. The alternating sum in Conjecture $\mathbf{VO}(X, n)$ is the so-called **secondary Euler characteristic**

$$\chi'(R\Gamma_{W,c}(X, \mathbb{Z}(n))) := \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)).$$

The calculations below show that the usual Euler characteristic of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ vanishes, assuming Conjectures $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X, n)$. See [41] for more details on the secondary Euler characteristic and its occurrences in nature.

Under the regulator conjecture, our vanishing order formula takes the form of the usual Euler characteristic of equivariant cohomology $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))$ or motivic cohomology $R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))[1]$.

Proposition 3.4. *Assuming* $\mathbf{L}^c(X_{\acute{e}t}, n)$ *and* $\mathbf{B}(X, n)$*, Conjecture* $\mathbf{VO}(X, n)$ *is equivalent to*

$$\begin{aligned} \operatorname{ord}_{s=n} \zeta(X, s) = \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) &= \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} \\ = -\chi(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))) &= \sum_{i \in \mathbb{Z}} (-1)^{i+1} \operatorname{rk}_{\mathbb{Z}} H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)). \end{aligned}$$

Moreover, we have

$$\chi(R\Gamma_{W,c}(X, \mathbb{Z}(n))) = 0.$$

Proof. Thanks to [2, Proposition 7.13], the Weil-étale complex tensored with $\mathbb{R}$ splits as

$$R\Gamma_{W,c}(X, \mathbb{R}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1].$$

Assuming Conjecture $\mathbf{B}(X, n)$, we also have a quasi-isomorphism

$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}),$$

so that

$$\dim_{\mathbb{R}} H^i_{W,c}(X, \mathbb{R}(n)) = \dim_{\mathbb{R}} H^{i-1}_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} + \dim_{\mathbb{R}} H^{i-2}_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}}.$$

Thus, we can rewrite the sum

$$\begin{aligned}\sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)) &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_{W,c}(X, \mathbb{R}(n)) \\ &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^{i-1}_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} \\ &\qquad + \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^{i-2}_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} \\ &= -\sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^{i-1}_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} \\ &= \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)) &= \chi(R\operatorname{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[1]) \\ &= -\chi(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))).\end{aligned}$$

These considerations also show that the usual Euler characteristic of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ vanishes. □

Remark 3.5. Conjecture $\mathbf{VO}(X, n)$ is related to a conjecture of Soulé [45, Conjecture 2.2], which originally reads in terms of K'-theory

$$\operatorname{ord}_{s=n} \zeta(X, s) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}} K'_i(X)_{(n)}.$$

As explained in [24, Remark 43], this can be rewritten in terms of Borel–Moore motivic homology as

$$\sum_{i\in\mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}} H^{BM}_i(X, \mathbb{Q}(n)).$$

In our setting, $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ plays the role of Borel–Moore homology, which explains the formula

$$\operatorname{ord}_{s=n} \zeta(X, s) = \sum_{i\in\mathbb{Z}} (-1)^{i+1} \operatorname{rk}_{\mathbb{Z}} H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)).$$

Remark 3.6 ([11, Proposition 5.13]). As for the formula

$$\operatorname{ord}_{s=n} \zeta(X, s) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_c(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}},$$

it essentially means that the vanishing order at $s = n < 0$ comes from the Archimedean Γ-factor appearing in the (hypothetical) functional equation, as explained in [43, §§3,4] (see also [12, §4]).

Indeed, under the assumption that $X_{\mathbb{C}}$ is a smooth projective variety, we consider the Hodge decomposition

$$H^i(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q},$$

which carries an action of $G_{\mathbb{R}} = \{id, \sigma\}$ such that $\sigma(H^{p,q}) = H^{q,p}$. We set $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$. For $p = i/2$ we consider the eigenspace decomposition $H^{p,p} = H^{p,+} \oplus H^{p,-}$, where

$$\begin{aligned} H^{p,+} &= \{x \in H^{p,p} \mid \sigma(x) = (-1)^p\, x\}, \\ H^{p,-} &= \{x \in H^{p,p} \mid \sigma(x) = (-1)^{p+1}\, x\}, \end{aligned}$$

and set $h^{p,\pm} = \dim_{\mathbb{C}} H^{p,\pm}$ accordingly. The completed zeta function

$$\zeta(\overline{X}, s) = \zeta(X, s)\, \zeta(X_\infty, s)$$

is expected to satisfy a functional equation of the form

$$A^{\frac{d-s}{2}}\, \zeta(\overline{X}, d-s) = A^{\frac{s}{2}}\, \zeta(\overline{X}, s).$$

Here

$$\begin{gathered} \zeta(X_\infty, s) = \prod_{i \in \mathbb{Z}} L_\infty(H^i(X), s)^{(-1)^i}, \\ L_\infty(H^i(X), s) = \prod_{p=i/2} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}}\, \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}} \prod_{\substack{p+q=i \\ p<q}} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}}, \\ \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\, \Gamma(s/2), \quad \Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\, \Gamma(s). \end{gathered}$$

Therefore, the expected vanishing order at $s = n < 0$ is

$$\begin{aligned} \operatorname{ord}_{s=n} \zeta(X, s) &= -\operatorname{ord}_{s=n} \zeta(X_\infty, s) \\ &= -\sum_{i \in \mathbb{Z}} (-1)^i \operatorname{ord}_{s=n} L_\infty(H^i(X), s) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \Bigl(\sum_{p=i/2} h^{p,(-1)^{n-p}} + \sum_{\substack{p+q=i \\ p<q}} h^{p,q} \Bigr). \end{aligned}$$

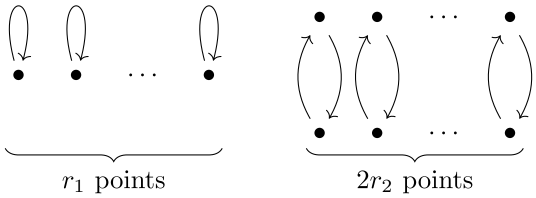
The last equality follows from the fact that $\Gamma(s)$ has simple poles at all $s = n \leq 0$. We have

$$\begin{aligned} \dim_{\mathbb{R}} H^i(X(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}} &= \dim_{\mathbb{R}} H^i(X(\mathbb{C}), \mathbb{R})^{\sigma = (-1)^n} \\ &= \dim_{\mathbb{C}} H^i(X(\mathbb{C}), \mathbb{C})^{\sigma = (-1)^n} \\ &= \sum_{p=i/2} h^{p,(-1)^{n-p}} + \sum_{\substack{p+q=i \\ p<q}} h^{p,q}. \end{aligned}$$

Here the terms $h^{p,q}$ with $p < q$ come from $\sigma(H^{p,q}) = H^{q,p}$, while $h^{p,(-1)^{n-p}}$ come from the action on $H^{p,p}$. We see that our conjectural formula recovers the expected vanishing order.

Let us look at some particular examples when the meromorphic continuation for $\zeta(X,s)$ is known.

Example 3.7. Suppose that $X = \operatorname{Spec} \mathcal{O}_F$ is the spectrum of the ring of integers of a number field $F/\mathbb{Q}$. Let r_1 be the number of real embeddings $F \hookrightarrow \mathbb{R}$ and r_2 be the number of conjugate pairs of complex embeddings $F \hookrightarrow \mathbb{C}$. The space $X(\mathbb{C})$ with the action of complex conjugation can be visualized as follows:



The complex $R\Gamma_c(X(\mathbb{C}),\mathbb{R}(n))$ consists of a single $G_\mathbb{R}$-module in degree 0 given by

$$\mathbb{R}(n)^{\oplus r_1} \oplus (\mathbb{R}(n) \oplus \mathbb{R}(n))^{\oplus r_2},$$

with the action of $G_\mathbb{R}$ on the first summand $\mathbb{R}(n)^{\oplus r_1}$ via the complex conjugation and the action on the second summand $(\mathbb{R}(n) \oplus \mathbb{R}(n))^{\oplus r_2}$ via $(x,y) \mapsto (\overline{y},\overline{x})$. The corresponding real space of fixed points has dimension

$$\dim_\mathbb{R} H^0_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n)) = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}, \end{cases}$$

which indeed coincides with the vanishing order of the Dedekind zeta function $\zeta(X,s) = \zeta_F(s)$ at $s = n < 0$.

On the motivic cohomology side, for $n < 0$ the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finite, except for $i = -1$, where by [19, Proposition 4.14]

$$\operatorname{rk}_\mathbb{Z} H^{-1}(X_{\acute{e}t}, \mathbb{Z}^c(n)) = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}. \end{cases}$$

Example 3.8. Suppose that X is a variety over a finite field $\mathbb{F}_q$. Then the vanishing order conjecture is not very interesting, because the formula yields

$$\begin{aligned} \operatorname{ord}_{s=n} \zeta(X,s) &= \sum_{i\in\mathbb{Z}} (-1)^i \dim_\mathbb{R} H^i_c(X(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}} \\ &= \sum_{i\in\mathbb{Z}} (-1)^{i+1} \operatorname{rk}_\mathbb{Z} H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0, \end{aligned}$$

since $X(\mathbb{C}) = \varnothing$, and also because $\mathbf{L}^c(X_{\acute{e}t}, n)$ implies $\operatorname{rk}_\mathbb{Z} H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0$ for all i in the case of varieties over finite fields, as observed in [2, Proposition 4.2]. Therefore, the conjecture simply asserts that $\zeta(X,s)$ has no zeros or poles at $s = n < 0$. This is indeed the case. We have $\zeta(X,s) = Z(X,q^{-s})$, where

$$Z(X,t) = \exp\Big(\sum_{k\ge 1} \frac{\#X(\mathbb{F}_{q^k})}{k}\, t^k\Big)$$

is the Hasse–Weil zeta function. According to Deligne's work on Weil's conjectures [8], the zeros and poles of $Z(X, s)$ satisfy $|s| = q^{-w/2}$, where $0 \leq w \leq 2 \dim X$ (see e.g. [26, pp. 26–27]). In particular, q^{-s} for $s = n < 0$ is neither a zero nor a pole of $Z(X, s)$.

We also note that our definition of $H^i_{W,c}(X, \mathbb{Z}(n))$, and pretty much everything said above, only makes sense for $n < 0$. Already for $n = 0$, for example, the zeta function of a smooth projective curve $X/\mathbb{F}_q$ has a simple pole at $s = 0$.

Example 3.9. Let $X = E$ be an integral model of an elliptic curve over $\mathbb{Q}$. Then, as a consequence of the modularity theorem (Wiles–Breuil–Conrad–Diamond–Taylor), it is known that $\zeta(E, s)$ admits a meromorphic continuation satisfying the functional equation with the Γ-factors discussed in Remark 3.6. In this particular case $\operatorname{ord}_{s=n} \zeta(E, s) = 0$ for all $n < 0$. This is consistent with the fact that $\chi(R\Gamma_c(G_\mathbb{R}, E(\mathbb{C}), \mathbb{R}(n))) = 0$.

Indeed, the equivariant cohomology groups $H^i_c(E(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}}$ are the following:

	$i = 0$	$i = 1$	$i = 2$
n even:	$\mathbb{R}$	$\mathbb{R}$	0
n odd:	0	$\mathbb{R}$	$\mathbb{R}$

—see, for example, the calculation in [44, Lemma A.6].

4 Special value Conjecture $\mathbf{C}(X, n)$

Definition 4.1. We define a morphism of complexes

$$\smile \theta\colon R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \otimes \mathbb{R}$$

using the splitting [2, Proposition 7.13]

$$R\Gamma_{W,c}(X, \mathbb{R}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1]$$

as follows:

$$\begin{tikzcd}
R\Gamma_{W,c}(X, \mathbb{R}(n)) \arrow[r, dashed, "\smile\theta"] \arrow[d, "\cong"] & R\Gamma_{W,c}(X, \mathbb{R}(n))[1] \arrow[d, "\cong"] \\
R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})[-1] & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R}) \\
\oplus & \oplus \\
R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[-1] \arrow[uur, "Reg^\vee_{X,n}"] & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))
\end{tikzcd}$$

Lemma 4.2. *Assuming Conjectures* $\mathbf{L}^c(X_{ét}, n)$ *and* $\mathbf{B}(X, n)$, *the morphism* $\smile \theta$ *induces a long exact sequence of finite dimensional real vector spaces*

$$\cdots \to H^{i-1}_{W,c}(X, \mathbb{R}(n)) \xrightarrow{\smile\theta} H^i_{W,c}(X, \mathbb{R}(n)) \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X, \mathbb{R}(n)) \to \cdots$$

Proof. We obtain a sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_{W,c}(X,\mathbb{R}(n)) \arrow[r, dashed, "\smile\theta"] \arrow[d, "\cong"] & H^{i+1}_{W,c}(X,\mathbb{R}(n)) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots & \begin{array}{c}\mathrm{Hom}(H^{-i-1}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})\\ \oplus \\ H^{i-1}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))\end{array} \arrow[r, "\cong"] & \begin{array}{c}\mathrm{Hom}(H^{-i-2}(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})\\ \oplus \\ H^{i}_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n))\end{array} & \cdots
\end{tikzcd}$$

The diagonal arrows are isomorphisms according to $\mathbf{B}(X,n)$, so the sequence is exact. $\square$

The Weil-étale complex $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ is defined in [2, §7] up to a *non-unique* isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$ via a distinguished triangle

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \xrightarrow{i_\infty} R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \to [1] \qquad (4.1)$$

This is rather awkward, and there should be a better, more canonical construction of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$. For our purposes, however, this is not much of a problem, since the special value conjecture is not formulated in terms of $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, but in terms of its determinant $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ (see Appendix A), which is well-defined.

Lemma 4.3. *The determinant* $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ *is defined up to a canonical isomorphism.*

Proof. Two different choices for the mapping fiber in (4.1) yield an isomorphism of distinguished triangles

$$\begin{tikzcd}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d, "f", "\cong"'] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i_\infty"] \arrow[d, "id"] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d, "id"] & {[1]} \arrow[d, "f", "\cong"'] \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))' \arrow[r] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i_\infty"] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] & {[1]}
\end{tikzcd}$$

The idea is to use functoriality of determinants with respect to isomorphisms of distinguished triangles (see Appendix A). The only technical problem is that whenever $X(\mathbb{R}) \neq \varnothing$, the complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$ and $R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))$ are not perfect, but may have finite 2-torsion in $H^i(-)$ for arbitrarily big i (in [2] we called such complexes **almost perfect**). On the other hand, the determinants are defined only for perfect complexes. Fortunately, $H^i(i^*_\infty)$ is an isomorphism for $i \gg 0$ by the boundedness of $H^i_{W,c}(X,\mathbb{Z}(n))$ [2, Proposition 7.12], so that for m big enough we can take the corresponding canonical truncations $\tau_{\leq m}$:

$$\begin{tikzcd}
\tau_{\leq m}R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & \tau_{\leq m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d] & \tau_{\leq m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & {[1]} \arrow[d, "\cong"] \\
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "i_\infty"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & {[1]} \arrow[d] \\
0 \arrow[r] \arrow[d] & \tau_{\geq m+1}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "\cong"] \arrow[d] & \tau_{\geq m+1}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
{[1]} \arrow[r] & {[1]} \arrow[r] & {[1]} \arrow[r] & {[2]}
\end{tikzcd}$$

The truncations give us (rotating the triangles)

$$\begin{tikzcd}
\tau_{\leq m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[-1] \arrow[r] \arrow[d, "id"] & R\Gamma_{W,c}(X, \mathbb{Z}(n)) \arrow[r] \arrow[d, "f", "\cong"'] & \tau_{\leq m} R\Gamma_{fg}(X, \mathbb{Z}(n)) \arrow[r] \arrow[d, "id"] & {[0]} \arrow[d, "id"] \\
\tau_{\leq m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[-1] \arrow[r] & R\Gamma_{W,c}(X, \mathbb{Z}(n))' \arrow[r] & \tau_{\leq m} R\Gamma_{fg}(X, \mathbb{Z}(n)) \arrow[r] & {[0]}
\end{tikzcd}$$

By the functoriality of determinants with respect to isomorphisms of distinguished triangles (see Appendix A), we have a commutative diagram

$$\begin{tikzcd}
\begin{array}{c} \det_{\mathbb{Z}} \tau_{\leq m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[-1] \\ \otimes \\ \det_{\mathbb{Z}} \tau_{\leq m} R\Gamma_{fg}(X, \mathbb{Z}(n)) \end{array} \arrow[r, "i", "\cong"'] \arrow[d, "id"] & \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \arrow[d, "\det_{\mathbb{Z}}(f)", "\cong"'] \\
\begin{array}{c} \det_{\mathbb{Z}} \tau_{\leq m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[-1] \\ \otimes \\ \det_{\mathbb{Z}} \tau_{\leq m} R\Gamma_{fg}(X, \mathbb{Z}(n)) \end{array} \arrow[r, "i'", "\cong"'] & \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))
\end{tikzcd}$$

so that $\det_{\mathbb{Z}}(f) = i' \circ i^{-1}$. ☐

LEMMA 4.4. *The non-canonical splitting*

$$R\Gamma_{W,c}(X, \mathbb{R}(n)) \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1]$$

from [2, Proposition 7.13] yields a canonical isomorphism of determinants

$$\det_{\mathbb{R}} R\Gamma_{W,c}(X, \mathbb{R}(n)) \cong \begin{array}{c} \det_{\mathbb{R}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \otimes_{\mathbb{R}} \\ \det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array}$$

Proof. This is similar to the previous lemma; in fact, after tensoring with $\mathbb{R}$, we obtain perfect complexes of real vector spaces, so the truncations are no longer needed. By [2, Proposition 7.4] we have $i_\infty^* \otimes \mathbb{R} = 0$, so there is an isomorphism of triangles

$$\begin{tikzcd}
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \arrow[r, "id"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \arrow[d] \\
R\Gamma_{W,c}(X, \mathbb{R}(n)) \arrow[r, dashed, "f", "\cong"'] \arrow[d] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[d] \\
R\Gamma_{fg}(X, \mathbb{R}(n)) \arrow[r, "g \otimes \mathbb{R}", "\cong"'] \arrow[d] & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \arrow[d] \\
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \arrow[r, "id"] & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))
\end{tikzcd} \tag{4.2}$$

Here the third horizontal arrow comes from the triangle defining $R\Gamma_{fg}(X, \mathbb{Z}(n))$:

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-2] \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{g} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1]$$

tensored with $\mathbb{R}$ (see [2, Proposition 5.7]). The distinguished column on the right-hand side of (4.2) is the direct sum

$$\begin{tikzcd}
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \arrow[d, "id"] & & 0 \arrow[d] \\
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \arrow[d] & \oplus & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \arrow[d, "id"] \\
0 \arrow[d] & & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \arrow[d] \\
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) & & 0
\end{tikzcd}$$

The splitting isomorphism f in (4.2) is not canonical at all. However, after taking the determinants, we obtain a commutative diagram (see Appendix A)

$$\begin{tikzcd}
\begin{array}{c} \det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \\ \otimes_{\mathbb{R}} \\ \det_{\mathbb{R}} R\Gamma_{fg}(X, \mathbb{R}(n)) \end{array} \arrow[r, "i", "\cong"'] \arrow[d, "\cong"', "id\otimes\det_{\mathbb{R}}(g\otimes\mathbb{R})"] & \det_{\mathbb{R}} R\Gamma_{W,c}(X, \mathbb{R}(n)) \arrow[d, "\cong"', "\det_{\mathbb{R}}(f)"] \arrow[dl, dashed, "\cong"] \\
\begin{array}{c} \det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \\ \otimes_{\mathbb{R}} \\ \det_{\mathbb{R}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \end{array} \arrow[r, "i'", "\cong"'] & \det_{\mathbb{R}} \begin{pmatrix} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{pmatrix}
\end{tikzcd}$$

The dashed diagonal arrow is the desired canonical isomorphism. □

Definition 4.5. Given an arithmetic scheme X and $n < 0$, assume Conjectures $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X, n)$. Consider the quasi-isomorphism

$$\begin{pmatrix} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{pmatrix} \xrightarrow[\cong]{Reg^{\vee}_{X,n}[-1]\oplus id} \begin{pmatrix} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{pmatrix}$$

$$\xrightarrow[\cong]{\text{split}} R\Gamma_{W,c}(X, \mathbb{R}(n)) \qquad (4.3)$$

Note that the first complex has determinant

$$\det_{\mathbb{R}} \begin{pmatrix} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{pmatrix} \cong \begin{array}{c} \det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \\ \otimes_{\mathbb{R}} \\ (\det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)))^{-1} \end{array} \cong \mathbb{R},$$

and for the last complex in (4.3), by the compatibility with base change, we have a canonical isomorphism

$$\det_{\mathbb{R}} R\Gamma_{W,c}(X, \mathbb{R}(n)) \cong (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes \mathbb{R}.$$

Therefore, after taking the determinants, the quasi-isomorphism (4.3) induces a canonical isomorphism

$$\lambda = \lambda_{X,n} \colon \mathbb{R} \xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes \mathbb{R}. \qquad (4.4)$$

Remark 4.6. An equivalent way to define λ is

$$\begin{aligned}\lambda\colon \mathbb{R} &\xrightarrow{\cong} \bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{R}} H^i_{W,c}(X,\mathbb{R}(n)))^{(-1)^i} \\ &\xrightarrow{\cong} \Big(\bigotimes_{i\in\mathbb{Z}} (\det_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)))^{(-1)^i}\Big) \otimes \mathbb{R} \\ &\xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes \mathbb{R},\end{aligned}$$

where the first isomorphism comes from Lemma 4.2.

We are ready to state the main conjecture of this paper. The determinant $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)))$ is a free $\mathbb{Z}$-module of rank 1, and the isomorphism (4.4) canonically embeds it in $\mathbb{R}$. We conjecture that this embedding gives the special value of $\zeta(X,s)$ at $s=n$ in the following sense.

Conjecture 4.7. $\mathbf{C}(X,n)$: let X be an arithmetic scheme and $n<0$ a strictly negative integer. Assuming Conjectures $\mathbf{L}^c(X_{\acute{e}t},n)$, $\mathbf{B}(X,n)$ and the meromorphic continuation of $\zeta(X,s)$ around $s=n<0$, the corresponding special value is determined up to sign by

$$\lambda(\zeta^*(X,n)^{-1})\cdot\mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)),$$

where λ is the canonical isomorphism (4.4).

Remark 4.8. This conjecture is similar to [11, Conjecture 5.12]. When X is proper and regular, the above conjecture is the same as the special value conjecture of Flach and Morin, which for $n\in\mathbb{Z}$ reads

$$\lambda_\infty\Big(\zeta^*(X,n)^{-1}\cdot C(X,n)\cdot\mathbb{Z}\Big) = \Delta(X/\mathbb{Z},n). \tag{4.5}$$

Here the fundamental line $\Delta(X/\mathbb{Z},n)$ is defined via

$$\Delta(X/\mathbb{Z},n) := \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \det_{\mathbb{Z}} R\Gamma_{dR}(X/\mathbb{Z})/\operatorname{Fil}^n.$$

If $n<0$, then $\Delta(X/\mathbb{Z},n) = \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$. Moreover, $C(X,n)$ in (4.5) is a rational number, defined via $\prod_p |c_p(X,n)|_p$. Here $c_p(X,n)\in\mathbb{Q}_p^\times/\mathbb{Z}_p^\times$ are the local factors described in [11, §5.4], but [11, Proposition 5.8] states that if $n\le 0$, then $c_p(X,n)\equiv 1 \pmod{\mathbb{Z}_p^\times}$ for all p. Therefore, $C(X,n)=1$ in our situation. Finally, the trivialization isomorphism λ_∞ is defined exactly as our λ. Therefore, (4.5) for $n<0$ agrees with Conjecture $\mathbf{C}(X,n)$.

Flach and Morin prove that their conjecture is consistent with the Tamagawa number conjecture of Bloch–Kato–Fontaine–Perrin-Riou [13]; see [11, §5.6] for the details.

Remark 4.9. Some canonical isomorphisms of determinants involve multiplication by ±1, so it is no surprise that the resulting conjecture is stated up to sign ±1. This is not a major problem, however, since the sign can be recovered from the (conjectural) functional equation.

5 Case of varieties over finite fields

For varieties over finite fields, our special value conjecture corresponds to the conjectures studied by Geisser in [15, 16, 17].

Proposition 5.1. *If $X/\mathbb{F}_q$ is a variety over a finite field, then under the assumption $\mathbf{L}^c(X_{ét}, n)$, the special value conjecture $\mathbf{C}(X, n)$ is equivalent to*

$$\begin{aligned} \zeta^*(X,n) &= \pm \prod_{i\in\mathbb{Z}} |H^i_{W,c}(X,\mathbb{Z}(n))|^{(-1)^i} \\ &= \pm \prod_{i\in\mathbb{Z}} |H^i(X_{ét},\mathbb{Z}^c(n))|^{(-1)^i} \\ &= \pm \prod_{i\in\mathbb{Z}} |H^c_i(X_{ar},\mathbb{Z}(n))|^{(-1)^{i+1}}, \end{aligned} \tag{5.1}$$

where $H^c_i(X_{ar}, \mathbb{Z}(n))$ are Geisser's arithmetic homology groups defined in [17].

Proof. Assuming $\mathbf{L}^c(X_{ét}, n)$, we have, thanks to [2, Proposition 7.7]

$$H^i_{W,c}(X,\mathbb{Z}(n)) \cong \operatorname{Hom}(H^{2-i}(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(H^c_{i-1}(X_{ar},\mathbb{Z}(n)),\mathbb{Q}/\mathbb{Z}).$$

The cohomology groups involved are finite and vanish for $|i| \gg 0$ by [2, Proposition 4.2], and by Lemma A.5 the determinant is given by

$$\begin{array}{ccc} \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) & \subset & \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{Q} \\ \parallel & & \parallel \\ \frac{1}{m}\mathbb{Z} & \subset & \mathbb{Q} \end{array}$$

where

$$m = \prod_{i\in\mathbb{Z}} |H^i_{W,c}(X,\mathbb{Z}(n))|^{(-1)^i}. \qquad \square$$

Remark 5.2. Formulas like (5.1) were proposed by Lichtenbaum early on in [33].

Theorem 5.3. *Let $X/\mathbb{F}_q$ be a variety over a finite field satisfying Conjecture $\mathbf{L}^c(X_{ét}, n)$ for $n < 0$. Then Conjecture $\mathbf{C}(X, n)$ holds.*

We note that (5.1) is equivalent to the special value formula that appears in [17, Theorem 4.5]. Conjecture $\mathbf{P}_0(X)$ in the statement of [17, Theorem 4.5] is implied by our Conjecture $\mathbf{L}^c(X_{ét}, n)$ thanks to [17, Proposition 4.1]. Geisser's proof eventually reduces to Milne's work [38], but for our case of $s = n < 0$, the situation is simpler, and we can give a direct explanation, using earlier results of Báyer and Neukirch [1] concerning Grothendieck's trace formula.

Proof. By the previous proposition, the conjecture reduces to

$$\zeta(X,n) = \prod_{i\in\mathbb{Z}} |H^i(X_{ét},\mathbb{Z}^c(n))|^{(-1)^i}.$$

By duality [2, Theorem I]

$$|H^{2-i}(X_{ét},\mathbb{Z}^c(n))| = |H^i_c(X_{ét},\mathbb{Z}(n))|,$$

where

$$\mathbb{Z}(n) := \bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)[-1] := \bigoplus_{\ell \neq p} \mu_{\ell^\infty}^{\otimes n}[-1] := \bigoplus_{\ell \neq p} \varinjlim_r \mu_{\ell^r}^{\otimes n}[-1],$$

and p is the characteristic of the base field. Now $H^i_c(X_{\acute{e}t}, \mathbb{Q}_\ell(n)) = 0$ for $n < 0$, and therefore $H^i_c(X_{\acute{e}t}, \mathbb{Z}_\ell(n)) \cong H^{i-1}_c(X_{\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. This means that our formula can be written as

$$\zeta(X, n) = \prod_{\ell \neq p} \prod_{i \in \mathbb{Z}} |H^i_c(X_{\acute{e}t}, \mathbb{Z}_\ell(n))|^{(-1)^i}. \tag{5.2}$$

Grothendieck's trace formula (see [21] or [7, Rapport]) reads

$$Z(X, t) = \prod_{i \in \mathbb{Z}} \det\bigl(1 - tF \bigm| H^i_c(\overline{X}, \mathbb{Q}_\ell)\bigr)^{(-1)^{i+1}},$$

where $\overline{X} := X \times_{\operatorname{Spec} \mathbb{F}_q} \overline{\mathbb{F}}_q$ and F is the Frobenius acting on $H^i_c(\overline{X}, \mathbb{Q}_\ell)$. Substituting $t = q^{-n}$,

$$\zeta(X, n) = \prod_{i \in \mathbb{Z}} \det\bigl(1 - q^{-n} F \bigm| H^i_c(\overline{X}, \mathbb{Q}_\ell)\bigr)^{(-1)^{i+1}}.$$

Then, by the proof of [1, Theorem (3.1)], for each $\ell \neq p$, we obtain

$$|\zeta(X, n)|_\ell = \prod_{i \in \mathbb{Z}} |H^i_c(X, \mathbb{Z}_\ell(n))|^{(-1)^{i+1}}. \tag{5.3}$$

On the other hand, for $n < 0$ we have

$$|\zeta(X, n)|_p = 1. \tag{5.4}$$

This fact can be justified, without assuming that X is smooth or projective, e.g., using Kedlaya's trace formula for rigid cohomology [27, p. 1446], which gives

$$Z(X, t) = \prod_i P_i(t)^{(-1)^{i+1}}, \quad \text{where } P_i(t) \in \mathbb{Z}[t] \text{ and } P_i(0) = 1.$$

In particular, $P_i(q^{-n}) \equiv 1 \pmod{p}$.

The product formula recovers from (5.3) and (5.4) our special value formula (5.2). ☐

Remark 5.4. The fact that $|\zeta(X, n)|_p = 1$, as observed in the argument above, explains why our Weil-étale cohomology ignores the p-primary part in some sense.

Let us consider a few examples to see how the special value conjecture works over finite fields.

Example 5.5. If $X = \operatorname{Spec} \mathbb{F}_q$, then $\zeta(X, s) = \frac{1}{1 - q^{-s}}$. In this case for $n < 0$ we obtain

$$H^i(\operatorname{Spec} \mathbb{F}_{q,\acute{e}t}, \mathbb{Z}^c(n)) \cong \begin{cases} \mathbb{Z}/(q^{-n} - 1), & i = 1, \\ 0, & i \neq 1 \end{cases} \tag{5.5}$$

(see, for example, [19, Example 4.2]). Therefore, formula (5.1) indeed recovers $\zeta(X, n)$ up to sign.

Similarly, if we replace $\operatorname{Spec} \mathbb{F}_q$ with $\operatorname{Spec} \mathbb{F}_{q^m}$, considered as a variety over $\mathbb{F}_q$, then $\zeta(\operatorname{Spec} \mathbb{F}_{q^m}, s) = \zeta(\operatorname{Spec} \mathbb{F}_q, ms)$, and (5.5) also changes accordingly.

Example 5.6. Consider $X = \mathbb{P}^1_{\mathbb{F}_q}/(0 \sim 1)$, or equivalently, a nodal cubic. The zeta function is $\zeta(X,s) = \frac{1}{1-q^{1-s}}$. We can calculate the groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ using the blowup square

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{F}_q \sqcup \operatorname{Spec} \mathbb{F}_q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{P}^1_{\mathbb{F}_q} \arrow[d] \\
\operatorname{Spec} \mathbb{F}_q \arrow[r] & X
\end{tikzcd}$$

This is similar to [16, §8, Example 2]. Geisser uses the eh-topology and long exact sequences associated to abstract blowup squares [16, Proposition 3.2]. In our case the same reasoning works, because by [2, Theorem I], one has $H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \operatorname{Hom}(H^{2-i}_c(X_{ét}, \mathbb{Z}(n)), \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}(n) = \varinjlim_{p \nmid m} \mu_m^{\otimes n}[-1]$, and étale cohomology and eh-cohomology coincide for such sheaves by [16, Theorem 3.6].

Using the projective bundle formula, we calculate from (5.5)

$$H^i(\mathbb{P}^1_{\mathbb{F}_q, ét}, \mathbb{Z}^c(n)) \cong \begin{cases} \mathbb{Z}/(q^{1-n}-1), & i = -1, \\ \mathbb{Z}/(q^{-n}-1), & i = +1, \\ 0, & i \neq \pm 1. \end{cases}$$

By the argument from [16, §8, Example 2], the short exact sequences

$$0 \to H^i(\mathbb{P}^1_{\mathbb{F}_q, ét}, \mathbb{Z}^c(n)) \to H^i(X_{ét}, \mathbb{Z}^c(n)) \to H^{i+1}((\operatorname{Spec} \mathbb{F}_q)_{ét}, \mathbb{Z}^c(n)) \to 0$$

give

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \cong \begin{cases} \mathbb{Z}/(q^{1-n}-1), & i = -1, \\ \mathbb{Z}/(q^{-n}-1), & i = 0, 1, \\ 0, & \text{otherwise}. \end{cases}$$

The formula (5.1) gives the correct value $\zeta(X,n)$.

Example 5.7. In general, if $X/\mathbb{F}_q$ is a curve, then Conjecture $\mathbf{L}^c(X_{ét}, n)$ holds; see for example [19, Proposition 4.3]. The cohomology $H^i(X_{ét}, \mathbb{Z}^c(n))$ is concentrated in degrees $-1, 0, +1$ by duality [2, Theorem I] and the reasons of cohomological dimension, and the special value formula is

$$\zeta^*(X,n) = \pm \frac{|H^0(X_{ét}, \mathbb{Z}^c(n))|}{|H^{-1}(X_{ét}, \mathbb{Z}^c(n))| \cdot |H^1(X_{ét}, \mathbb{Z}^c(n))|}.$$

6 Compatibility with operations on schemes

The following basic properties follow from the definition of $\zeta(X,s)$ (formula (1.1)).

1) **Disjoint unions**: if $X = \coprod_{1 \le i \le r} X_i$ is a finite disjoint union of arithmetic schemes, then

$$\zeta(X,s) = \prod_{1 \le i \le r} \zeta(X_i, s). \tag{6.1}$$

In particular,

$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{1\le i\le r} \operatorname{ord}_{s=n} \zeta(X_i,s),$$
$$\zeta^*(X,n) = \prod_{1\le i\le r} \zeta^*(X_i,n).$$

2) **Closed-open decompositions**: if $Z \subset X$ is a closed subscheme and $U = X \setminus Z$ is its open complement, then we say that we have a **closed-open decomposition** and write $Z \not\hookrightarrow X \hookleftarrow U$. In this case

$$\zeta(X,s) = \zeta(Z,s) \cdot \zeta(U,s). \tag{6.2}$$

In particular,

$$\operatorname{ord}_{s=n} \zeta(X,s) = \operatorname{ord}_{s=n} \zeta(Z,s) + \operatorname{ord}_{s=n} \zeta(U,s),$$
$$\zeta^*(X,n) = \zeta^*(Z,n) \cdot \zeta^*(U,n).$$

3) **Affine bundles**: for any $r \ge 0$ the zeta function of the relative affine space $\mathbb{A}^r_X = \mathbb{A}^r_{\mathbb{Z}} \times X$ satisfies

$$\zeta(\mathbb{A}^r_X, s) = \zeta(X, s-r). \tag{6.3}$$

In particular,

$$\operatorname{ord}_{s=n} \zeta(\mathbb{A}^r_X,s) = \operatorname{ord}_{s=n-r} \zeta(X,s),$$
$$\zeta^*(\mathbb{A}^r_X,n) = \zeta^*(X,n-r).$$

This suggests that Conjectures $\mathbf{VO}(X,n)$ and $\mathbf{C}(X,n)$ should also satisfy the corresponding compatibilities. We verify in this section that this is indeed the case.

Lemma 6.1. *Let $n < 0$.*

1) If $X = \coprod_{1\le i\le r} X_i$ is a finite disjoint union of arithmetic schemes, then

$$\mathbf{L}^c(X_{\acute{e}t},n) \iff \mathbf{L}^c(X_{i,\acute{e}t},n) \text{ for all } i.$$

2) For a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$, if two of the three conjectures

$$\mathbf{L}^c(X_{\acute{e}t},n), \quad \mathbf{L}^c(Z_{\acute{e}t},n), \quad \mathbf{L}^c(U_{\acute{e}t},n)$$

are true, then the third is also true.

3) For an arithmetic scheme X and any $r \ge 0$, one has

$$\mathbf{L}^c(\mathbb{A}^r_{X,\acute{e}t},n) \iff \mathbf{L}^c(X_{\acute{e}t},n-r).$$

Proof. See the proof of [39, Proposition 5.10]. ☐

Lemma 6.2. *Let $n < 0$.*

1) *If $X = \coprod_{1 \le i \le r} X_i$ is a finite disjoint union of arithmetic schemes, then*

$$Reg_{X,n} = \bigoplus_{1 \le i \le r} Reg_{X_i,n} \colon$$
$$\bigoplus_{1 \le i \le r} R\Gamma(X_{i,\acute{e}t}, \mathbb{R}^c(n)) \to \bigoplus_{i \le i \le r} R\Gamma_{BM}(G_\mathbb{R}, X_i(\mathbb{C}), \mathbb{R}(n))[1].$$

In particular,

$$\mathbf{B}(X,n) \iff \mathbf{B}(X_i,n) \text{ for all } i.$$

2) *For a closed-open decomposition of arithmetic schemes $Z \not\hookrightarrow X \hookleftarrow U$, the corresponding regulators give a morphism of distinguished triangles*

$$\begin{tikzcd}
R\Gamma(Z_{\acute{e}t}, \mathbb{R}^c(n)) \arrow[r, "Reg_{Z,n}"] \arrow[d] & R\Gamma_{BM}(G_\mathbb{R}, Z(\mathbb{C}), \mathbb{R}(n))[1] \arrow[d] \\
R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \arrow[r, "Reg_{X,n}"] \arrow[d] & R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n))[1] \arrow[d] \\
R\Gamma(U_{\acute{e}t}, \mathbb{R}^c(n)) \arrow[r, "Reg_{U,n}"] \arrow[d] & R\Gamma_{BM}(G_\mathbb{R}, U(\mathbb{C}), \mathbb{R}(n))[1] \arrow[d] \\
R\Gamma(Z_{\acute{e}t}, \mathbb{R}^c(n))[1] \arrow[r, "{Reg_{Z,n}[1]}"] & R\Gamma_{BM}(G_\mathbb{R}, Z(\mathbb{C}), \mathbb{R}(n))[2]
\end{tikzcd}$$

In particular, if two of the three conjectures

$$\mathbf{B}(X,n), \quad \mathbf{B}(Z,n), \quad \mathbf{B}(U,n)$$

are true, then the third is also true.

3) *For any $r \ge 0$, the diagram*

$$\begin{tikzcd}
R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n-r))[2r] \arrow[r, "\cong"] \arrow[d, "Reg_{X,n-r}"] & R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{R}^c(n)) \arrow[d, "Reg_{\mathbb{A}^r_X,n}"] \\
R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), \mathbb{R}(n-r))[2r] \arrow[r, "\cong"] & R\Gamma_{BM}(G_\mathbb{R}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))
\end{tikzcd}$$

commutes. In particular, one has

$$\mathbf{B}(\mathbb{A}^r_X, n) \iff \mathbf{B}(X, n-r).$$

Proof. Part 1) is clear because all cohomologies that enter the definition of $Reg_{X,n}$ decompose into direct sums over $i = 1, \ldots r$. Parts 2) and 3) boil down to the corresponding functoriality properties for the KLM morphism (2.1), namely that it commutes with proper pushforwards and flat pullbacks by [47, Lemma 3 and 4]. For closed-open decompositions, the distinguished triangle

$$R\Gamma(Z_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{R}^c(n)) \to R\Gamma(Z_{\acute{e}t}, \mathbb{R}^c(n))[1]$$

comes exactly from the proper pushforward along $Z \hookrightarrow X$ and flat pullback along $U \hookrightarrow X$ (see [18, Corollary 7.2] and [4, §3]). Similarly, the quasi-isomorphism $R\Gamma(X_{\acute{e}t}, \mathbb{R}^c(n-r))[2r] \cong R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{R}^c(n))$ results from the flat pullback along $p \colon \mathbb{A}^r_X \to X$. ☐

Proposition 6.3. *For each arithmetic scheme X below and $n < 0$, assume $\mathbf{L}^c(X_{ét}, n)$, $\mathbf{B}(X, n)$, and the meromorphic continuation of $\zeta(X, s)$ around $s = n$.*

1) *If $X = \coprod_{1 \le i \le r} X_i$ is a finite disjoint union of arithmetic schemes, then*
$$\mathbf{VO}(X, n) \iff \mathbf{VO}(X_i, n) \text{ for all } i.$$

2) *For a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$, if two of the three conjectures*
$$\mathbf{VO}(X, n), \quad \mathbf{VO}(Z, n), \quad \mathbf{VO}(U, n)$$
are true, then the third is also true.

3) *For any $r \ge 0$, one has*
$$\mathbf{VO}(\mathbb{A}^r_X, n) \iff \mathbf{VO}(X, n - r).$$

Proof. We have already observed in Proposition 3.4 that under Conjecture $\mathbf{B}(X, n)$ we can rewrite $\mathbf{VO}(X, n)$ as
$$\operatorname{ord}_{s=n} \zeta(X, s) = \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))).$$

In part 1), we have
$$\operatorname{ord}_{s=n} \zeta(X, s) = \sum_{1 \le i \le r} \operatorname{ord}_{s=n} \zeta(X_i, s),$$
and for the corresponding $G_{\mathbb{R}}$-equivariant cohomology,
$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) = \bigoplus_{1 \le i \le r} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)).$$
The statement follows from the additivity of the Euler characteristic:
$$\begin{tikzcd}
\operatorname{ord}_{s=n} \zeta(X, s) \arrow[r, equal, "\mathbf{VO}(X,n)"] \arrow[d, equal] & \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))) \arrow[d, equal] \\
\sum\limits_{1 \le i \le r} \operatorname{ord}_{s=n} \zeta(X_i, s) \arrow[r, equal, "\forall i \mathbf{VO}(X_i,n)"] & \sum\limits_{1 \le i \le r} \chi(R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n)))
\end{tikzcd}$$

Similarly in part 2), we can consider the distinguished triangle
$$R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \to R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n)) \\ \to R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[1]$$
and the additivity of the Euler characteristic gives
$$\begin{tikzcd}
\operatorname{ord}_{s=n} \zeta(X, s) \arrow[r, equal, "\mathbf{VO}(X,n)"] \arrow[d, equal] & \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))) \arrow[d, equal] \\
\operatorname{ord}_{s=n} \zeta(Z, s) \arrow[r, equal, "\mathbf{VO}(Z,n)"] & \chi(R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n))) \\
+ & + \\
\operatorname{ord}_{s=n} \zeta(U, s) \arrow[r, equal, "\mathbf{VO}(U,n)"] & \chi(R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)))
\end{tikzcd}$$

Finally, in part 3), we assume for simplicity that $X_{\mathbb{C}}$ is connected of dimension $d_{\mathbb{C}}$. Then the Poincaré duality and homotopy invariance of the usual cohomology without compact support give us

$$\begin{aligned}
&R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r(\mathbb{C}) \times X(\mathbb{C}), \mathbb{R}(n)) \\
&\quad \overset{\text{P.D.}}{\cong} R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, \mathbb{A}^r(\mathbb{C}) \times X(\mathbb{C}), \mathbb{R}(d_{\mathbb{C}} + r - n)), \mathbb{R})[-2d_{\mathbb{C}} - 2r] \\
&\qquad \overset{\text{H.I.}}{\cong} R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(d_{\mathbb{C}} + r - n)), \mathbb{R})[-2d_{\mathbb{C}} - 2r] \\
&\qquad\qquad\qquad\qquad \overset{\text{P.D.}}{\cong} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n - r))[-2r].
\end{aligned}$$

The twist $[-2r]$ is even and therefore has no effect on the Euler characteristic, so that we obtain

$$\begin{tikzcd}
\operatorname{ord}_{s=n} \zeta(\mathbb{A}^r_X, s) \arrow[r, equal, "\mathbf{VO}(\mathbb{A}^r_X, n)"] \arrow[d, equal] & \chi(R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r(\mathbb{C}) \times X(\mathbb{C}), \mathbb{R}(n))) \arrow[d, equal] \\
\operatorname{ord}_{s=n-r} \zeta(X, s) \arrow[r, equal, "\mathbf{VO}(X, n-r)"] & \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n - r)))
\end{tikzcd}$$

□

Our next goal is to prove similar compatibilities for Conjecture $\mathbf{C}(X, n)$, as was just done for $\mathbf{VO}(X, n)$. We split the proof into three technical lemmas 6.4, 6.5, 6.7, each for the corresponding compatibility.

Lemma 6.4. *Let $n < 0$ and let $X = \coprod_{1 \le i \le r} X_i$ be a finite disjoint union of arithmetic schemes. Assume $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X, n)$. Then there is a quasi-isomorphism of complexes*

$$\bigoplus_{1 \le i \le r} R\Gamma_{W,c}(X_i, \mathbb{Z}(n)) \cong R\Gamma_{W,c}(X, \mathbb{Z}(n)), \tag{6.4}$$

which after taking the determinants gives a commutative diagram

$$\begin{tikzcd}
\mathbb{R} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{R} \arrow[rr, "x_1 \otimes \cdots \otimes x_r \mapsto x_1 \cdots x_r", "\cong"'] \arrow[d, "\cong"', "\lambda_{X_1,n} \otimes \cdots \otimes \lambda_{X_r,n}"] & & \mathbb{R} \arrow[d, "\cong"', "\lambda_{X,n}"] \\
\bigotimes_{1 \le i \le r} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X_i, \mathbb{Z}(n))) \otimes \mathbb{R} \arrow[rr, "\cong"] & & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X_i, \mathbb{Z}(n))) \otimes \mathbb{R}
\end{tikzcd} \tag{6.5}$$

Proof. For $X = \coprod_{1 \le i \le r} X_i$, all cohomologies in our construction of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ in [2] decompose into the corresponding direct sum over $i = 1, \ldots, r$, and (6.4) follows.

After tensoring with $\mathbb{R}$, we obtain a commutative diagram

$$\begin{tikzcd}
\bigoplus_i \begin{pmatrix} R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n))[-1] \end{pmatrix} \arrow[r, "\cong"] \arrow[d, "\cong"', "\bigoplus_i Reg^{\vee}_{X_i,n}{[-1]}\oplus id"] & \begin{matrix} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{matrix} \arrow[d, "\cong"', "Reg^{\vee}_{X,n}{[-1]}\oplus id"] \\
\bigoplus_i \begin{pmatrix} R\mathrm{Hom}(R\Gamma(X_{i,\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X_i(\mathbb{C}), \mathbb{R}(n))[-1] \end{pmatrix} \arrow[r, "\cong"] \arrow[d, "\cong"', "\mathrm{split}"] & \begin{matrix} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{matrix} \arrow[d, "\cong"', "\mathrm{split}"] \\
\bigoplus_i R\Gamma_{W,c}(X_i, \mathbb{R}(n)) \arrow[r, "\cong"] & R\Gamma_{W,c}(X, \mathbb{R}(n))
\end{tikzcd}$$

Taking the determinants, we obtain (6.5). ☐

Lemma 6.5. *Let $n < 0$ and let $Z \not\hookrightarrow X \hookleftarrow U$ be a closed-open decomposition of arithmetic schemes, such that the conjectures*

$$\mathbf{L}^c(U_{\acute{e}t}, n),\ \mathbf{L}^c(X_{\acute{e}t}, n),\ \mathbf{L}^c(Z_{\acute{e}t}, n),$$
$$\mathbf{B}(U, n),\ \mathbf{B}(X, n),\ \mathbf{B}(Z_{\acute{e}t}, n)$$

hold (it suffices to assume two of the three conjectures thanks to Lemmas 6.1 and 6.2). Then there is an isomorphism of determinants

$$\det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(U, \mathbb{Z}(n)) \otimes \det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(Z, \mathbb{Z}(n)) \cong \det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \tag{6.6}$$

making the following diagram commute up to signs:

$$\begin{tikzcd}
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \arrow[r, "x\otimes y\mapsto xy"] \arrow[d, "\cong"', "\lambda_{U,n}\otimes\lambda_{Z,n}"] & \mathbb{R} \arrow[d, "\cong"', "\lambda_{X,n}"] \\
\begin{array}{c} (\det_{\mathbb{Z}} R\Gamma_{W,c}(U, \mathbb{Z}(n))) \otimes \mathbb{R} \\ \otimes_{\mathbb{R}} \\ (\det_{\mathbb{Z}} R\Gamma_{W,c}(Z, \mathbb{Z}(n))) \otimes \mathbb{R} \end{array} \arrow[r, "\cong"] & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes \mathbb{R}
\end{tikzcd} \tag{6.7}$$

Proof. A closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$ induces the distinguished triangles

$$\begin{tikzcd}
R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \arrow[r] & R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \arrow[r] & R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \arrow[r] & {[1]} \\
R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r] & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r] & R\Gamma_c(Z_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r] & {[1]} \\
R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)) \arrow[r] & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)) \arrow[r] & R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n)) \arrow[r] & {[1]}
\end{tikzcd}$$

The first triangle is [18, Corollary 7.2] and it means that $R\Gamma(-, \mathbb{Z}^c(n))$ behaves like Borel–Moore homology. The following two are the usual triangles for cohomology with compact support. These fit together in a commutative diagram shown in Figure 1 below (p. 29). Figure 2 on p. 30 shows the same diagram tensored with $\mathbb{R}$.

In this diagram we start from the morphism of triangles $(\alpha_{U,n}, \alpha_{X,n}, \alpha_{Z,n})$ and then take the corresponding cones $R\Gamma_{fg}(-, \mathbb{Z}(n))$. By [2, Proposition 5.6], these cones are defined up to a *unique* isomorphism in the derived category $\mathbf{D}(\mathbb{Z})$, and the same argument shows that the induced morphisms of complexes

$$R\Gamma_{fg}(U,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(Z,\mathbb{Z}(n)) \to R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \quad (6.8)$$

are also well-defined (see [2, Corollary A.3]). A priori, (6.8) need not be a distinguished triangle, but we claim that it induces a long exact sequence in cohomology.

To this end, note that tensoring the diagram with $\mathbb{Z}/m\mathbb{Z}$ gives us an isomorphism

$$\begin{tikzcd}
R\Gamma_c(U_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}/m\mathbb{Z}(n)) \arrow[r] \arrow[d, "\cong"] & {[1]} \arrow[d, "\cong"] \\
\begin{array}{c} R\Gamma_{fg}(U,\mathbb{Z}(n)) \\ \otimes^{\mathbf{L}} \\ \mathbb{Z}/m\mathbb{Z} \end{array} \arrow[r] & \begin{array}{c} R\Gamma_{fg}(X,\mathbb{Z}(n)) \\ \otimes^{\mathbf{L}} \\ \mathbb{Z}/m\mathbb{Z} \end{array} \arrow[r] & \begin{array}{c} R\Gamma_{fg}(Z,\mathbb{Z}(n)) \\ \otimes^{\mathbf{L}} \\ \mathbb{Z}/m\mathbb{Z} \end{array} \arrow[r] & {[1]}
\end{tikzcd}$$

More generally, for each prime p we can take the corresponding derived p-adic completions (see [3] and [46, Tag 091N])

$$R\Gamma_{fg}(-,\mathbb{Z}(n))^{\wedge}_{p} := R\varprojlim_{k}(R\Gamma_{fg}(-,\mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/p^k\mathbb{Z}),$$

which give us a distinguished triangle for each prime p

$$R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_{p} \to R\Gamma_{fg}(X,\mathbb{Z}(n))^{\wedge}_{p} \to R\Gamma_{fg}(Z,\mathbb{Z}(n))^{\wedge}_{p} \to R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_{p}[1].$$

At the level of cohomology, there are natural isomorphisms [46, Tag 0A06]

$$H^i(R\Gamma_{fg}(-,\mathbb{Z}(n))^{\wedge}_{p}) \cong H^i_{fg}(-,\mathbb{Z}(n)) \otimes \mathbb{Z}_p.$$

In particular, for each p there is a long exact sequence of cohomology groups

$$\begin{aligned}\cdots \to H^i_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to H^i_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to H^i_{fg}(Z,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \\ \to H^{i+1}_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to \cdots\end{aligned}$$

induced by (6.8). By finite generation of $H^i_{fg}(-,\mathbb{Z}(n))$ and flatness of $\mathbb{Z}_p$ this implies that the sequence

$$\cdots \to H^i_{fg}(U,\mathbb{Z}(n)) \to H^i_{fg}(X,\mathbb{Z}(n)) \to H^i_{fg}(Z,\mathbb{Z}(n)) \to H^{i+1}_{fg}(U,\mathbb{Z}(n)) \to \cdots \quad (6.9)$$

is exact.

Now we consider the diagram

$$\begin{tikzcd}
\tau_{\le m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n))[-1] \arrow[r] \arrow[d] & R\Gamma_{W,c}(U,\mathbb{Z}(n)) \arrow[r] \arrow[d] & \tau_{\le m}R\Gamma_{fg}(U,\mathbb{Z}(n)) \arrow[r] \arrow[d] & {[0]} \arrow[d] \\
\tau_{\le m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n))[-1] \arrow[r] \arrow[d] & R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d] & \tau_{\le m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d] & {[0]} \arrow[d] \\
\tau_{\le m}R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{Z}(n))[-1] \arrow[r] \arrow[d] & R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \arrow[r] \arrow[d] & \tau_{\le m}R\Gamma_{fg}(Z,\mathbb{Z}(n)) \arrow[r] \arrow[d] & {[0]} \arrow[d] \\
\tau_{\le m}R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n)) \arrow[r] & R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] \arrow[r] & \tau_{\le m}R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \arrow[r] & {[1]}
\end{tikzcd}$$

Here we took truncations for m big enough, as in the proof of Lemma 4.3. There are canonical isomorphisms

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(U,\mathbb{Z}(n)) \cong \begin{array}{c} \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{Z}(n))[-1]) \\ \otimes \\ \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(U,\mathbb{Z}(n))), \end{array}$$

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \begin{array}{c} \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))[-1]) \\ \otimes \\ \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(X,\mathbb{Z}(n))), \end{array}$$

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \cong \begin{array}{c} \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{Z}(n))[-1]) \\ \otimes \\ \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(Z,\mathbb{Z}(n))), \end{array}$$

$$\det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n))) \cong \begin{array}{c} \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{Z}(n))) \\ \otimes \\ \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{Z}(n))), \end{array}$$

$$\det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(X,\mathbb{Z}(n))) \cong \begin{array}{c} \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(U,\mathbb{Z}(n))) \\ \otimes \\ \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(Z,\mathbb{Z}(n))). \end{array}$$

The first four isomorphisms arise from the corresponding distinguished triangles, while the last isomorphism comes from the long exact sequence (6.9), which gives an isomorphism

$$\bigotimes_{i\le m} \Bigl(\det_{\mathbb{Z}} H^i_{fg}(U,\mathbb{Z}(n))^{(-1)^i} \otimes \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n))^{(-1)^{i+1}} \otimes \det_{\mathbb{Z}} H^i_{fg}(Z,\mathbb{Z}(n))^{(-1)^i} \Bigr) \cong \mathbb{Z}. \tag{6.10}$$

We can rearrange the terms (at the expense of introducing a ±1 sign), to obtain

$$\begin{aligned} \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(X,\mathbb{Z}(n))) &\cong \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n)) \cong \\ &\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(U,\mathbb{Z}(n)) \otimes \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(Z,\mathbb{Z}(n)) \cong \\ &\det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(U,\mathbb{Z}(n))) \otimes \det_{\mathbb{Z}}(\tau_{\le m} R\Gamma_{fg}(Z,\mathbb{Z}(n))). \end{aligned}$$

All this gives us the desired isomorphism of integral determinants (6.6).

Let us now consider the diagram with distinguished rows in Figure 3 (p. 31). Here the three squares with the regulators involved commute thanks to Lemma 6.2. Taking the determinants, we obtain (6.7), by the compatibility with distinguished triangles. □

Remark 6.6. Morally, we expect that a closed-open decomposition induces a distinguished triangle of the form

$$R\Gamma_{W,c}(U,\mathbb{Z}(n)) \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \to [1]. \tag{6.11}$$

However, $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is defined in [2] as a mapping fiber of a morphism in $\mathbf{D}(\mathbb{Z})$, so it is not quite functorial.

We recall that in the usual derived (1-)category $\mathbf{D}(\mathcal{A})$, taking naively a "cone of a morphism of distinguished triangles"

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & A^\bullet[1] \arrow[d] \\
A^{\bullet\prime} \arrow[r] \arrow[d, dashed] & B^{\bullet\prime} \arrow[r] \arrow[d, dashed] & C^{\bullet\prime} \arrow[r] \arrow[d, dashed] & A^{\bullet\prime}[1] \arrow[d, dashed] \\
A^{\bullet\prime\prime} \arrow[r, dashed] \arrow[d, dashed] & B^{\bullet\prime\prime} \arrow[r, dashed] \arrow[d, dashed] & C^{\bullet\prime\prime} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "(-)"] & A^{\bullet\prime\prime}[1] \arrow[d, dashed] \\
A^\bullet[1] \arrow[r] & B^\bullet[1] \arrow[r] & C^\bullet[1] \arrow[r] & A^\bullet[2]
\end{tikzcd}$$

usually *does not* yield a distinguished triangle $A^{\bullet\prime\prime} \to B^{\bullet\prime\prime} \to C^{\bullet\prime\prime} \to A^{\bullet\prime\prime}[1]$. For a thorough discussion of this problem, see [40].

For lack of a better definition for $R\Gamma_{W,c}(X, \mathbb{Z}(n))$, we constructed the isomorphism (6.6) ad hoc, without the hypothetical triangle (6.11).

$$\begin{tikzcd}
 & & & & R\Gamma_{W,c}(U,\mathbb{Z}(n)) \arrow[dl] & \\
R\mathrm{Hom}(R\Gamma(U_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r, "\alpha_{U,n}"] \arrow[dd] & R\Gamma_c(U_{ét},\mathbb{Z}(n)) \arrow[rr] \arrow[dr, "u_\infty"] \arrow[dd] & & R\Gamma_{fg}(U,\mathbb{Z}(n)) \arrow[rr] \arrow[dl, "i_\infty"'] \arrow[dd, dashed, "\exists!"] & & \cdots[-1] \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n)) \arrow[dd] & & R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[dl] & \\
R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r, "\alpha_{X,n}"] \arrow[dd] & R\Gamma_c(X_{ét},\mathbb{Z}(n)) \arrow[rr] \arrow[dr, "u_\infty"] \arrow[dd] & & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[rr] \arrow[dl, "i_\infty"'] \arrow[dd, dashed, "\exists!"] & & \cdots[-1] \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{Z}(n)) \arrow[dd] & & R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \arrow[dl] & \\
R\mathrm{Hom}(R\Gamma(Z_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r, "\alpha_{Z,n}"] \arrow[dd] & R\Gamma_c(Z_{ét},\mathbb{Z}(n)) \arrow[rr] \arrow[dr, "u_\infty"] \arrow[dd] & & R\Gamma_{fg}(Z,\mathbb{Z}(n)) \arrow[rr] \arrow[dl, "i_\infty"'] \arrow[dd, dashed, "\exists!"] & & \cdots[-1] \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{Z}(n)) \arrow[dd] & & R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] \arrow[dl] & \\
R\mathrm{Hom}(R\Gamma(U_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \arrow[r, "{\alpha_{U,n}[1]}"] & R\Gamma_c(U_{ét},\mathbb{Z}(n))[1] \arrow[rr] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \arrow[rr] \arrow[dl, "{i_\infty[1]}"'] & & \cdots[0] \\
 & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{Z}(n))[1] & & &
\end{tikzcd}$$

Figure 1: Diagram induced by a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$.

$$\begin{tikzcd}
 & & & & R\Gamma_{W,c}(U,\mathbb{R}(n)) \arrow[dl] \arrow[dd] & \\
R\mathrm{Hom}(R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-2]) \arrow[r] \arrow[dd] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & R\Gamma_{fg}(U,\mathbb{R}(n)) \arrow[rr, "\cong"] \arrow[dl, "0"'] \arrow[dd, dashed, "\exists!"] & & R\mathrm{Hom}(R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1]) \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{R}(n)) \arrow[dd] & & R\Gamma_{W,c}(X,\mathbb{R}(n)) \arrow[dl] \arrow[dd] & \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-2]) \arrow[r] \arrow[dd] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & R\Gamma_{fg}(X,\mathbb{R}(n)) \arrow[rr, "\cong"] \arrow[dl, "0"'] \arrow[dd, dashed, "\exists!"] & & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1]) \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),\mathbb{R}(n)) \arrow[dd] & & R\Gamma_{W,c}(Z,\mathbb{R}(n)) \arrow[dl] \arrow[dd] & \\
R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-2]) \arrow[r] \arrow[dd] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & R\Gamma_{fg}(Z,\mathbb{R}(n)) \arrow[rr, "\cong"] \arrow[dl, "0"'] \arrow[dd, dashed, "\exists!"] & & R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1]) \arrow[dd] \\
 & & R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),\mathbb{R}(n)) \arrow[dd] & & R\Gamma_{W,c}(U,\mathbb{R}(n))[1] \arrow[dl] & \\
R\mathrm{Hom}(R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1]) \arrow[r] & 0 \arrow[rr] \arrow[dr] & & R\Gamma_{fg}(U,\mathbb{R}(n))[1] \arrow[rr, "\cong"] \arrow[dl, "0"'] & & R\mathrm{Hom}(R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}) \\
 & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),\mathbb{R}(n))[1] & & &
\end{tikzcd}$$

Figure 2: Diagram induced by a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$, tensored with $\mathbb{R}$.

$$\begin{tikzcd}
\begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r] \arrow[d, "\cong"', "Reg^{\vee}_{U,n}[-1]\oplus id"] & \begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r] \arrow[d, "\cong"', "Reg^{\vee}_{X,n}[-1]\oplus id"] & \begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r] \arrow[d, "\cong"', "Reg^{\vee}_{Z,n}[-1]\oplus id"] & \begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)) \end{array} \arrow[d, "\cong"', "Reg^{\vee}_{U,n}\oplus id"] \\
\begin{array}{c} R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r] \arrow[d, "\cong"', "\text{split}"] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r] \arrow[d, "\cong"', "\text{split}"] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r] \arrow[d, "\cong"', "\text{split}"] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), \mathbb{R}(n)) \end{array} \arrow[d, "\cong"', "\text{split}"] \\
R\Gamma_{W,c}(U, \mathbb{R}(n)) \arrow[r] & R\Gamma_{W,c}(X, \mathbb{R}(n)) \arrow[r] & R\Gamma_{W,c}(Z, \mathbb{R}(n)) \arrow[r] & R\Gamma_{W,c}(U, \mathbb{R}(n))[1]
\end{tikzcd}$$

Figure 3: Diagram induced by a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$

Lemma 6.7. *For $n < 0$ and $r \geq 0$, let X be an arithmetic scheme satisfying $\mathbf{L}^c(X_{ét}, n-r)$ and $\mathbf{B}(X, n-r)$. Then there is a natural quasi-isomorphism of complexes*

$$R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{Z}(n)) \cong R\Gamma_{W,c}(X, \mathbb{Z}(n-r))[-2r], \tag{6.12}$$

which after passing to the determinants makes the following diagram commute:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[dl, "\lambda_{\mathbb{A}^r_X,n}"', "\cong"] \arrow[dr, "\lambda_{X,n-r}", "\cong"'] & \\
(\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{Z}(n))) \otimes \mathbb{R} \arrow[rr, "\cong"] & & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n-r))) \otimes \mathbb{R}
\end{tikzcd} \tag{6.13}$$

Proof. We refer to Figure 4 (p. 33), which shows how the flat morphism $p\colon \mathbb{A}^r_X \to X$ induces the desired quasi-isomorphism (6.12). It all boils down to the homotopy property of motivic cohomology, namely the fact that p induces a quasi-isomorphism

$$p^*\colon R\Gamma(X_{ét}, \mathbb{Z}^c(n-r))[2r] \xrightarrow{\cong} R\Gamma(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n));$$

see, e.g. [39, Lemma 5.11]. After passing to real coefficients, we obtain the following diagram:

$$\begin{tikzcd}
\begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r, "\cong"] \arrow[d, "Reg^\vee_{\mathbb{A}^r_X,n}[-1]\oplus id", "\cong"'] & \begin{array}{c} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-2][-2r] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-1][-2r] \end{array} \arrow[d, "Reg^\vee_{X,n-r}[-1][-2r]\oplus id", "\cong"'] \\
\begin{array}{c} R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n))[-1] \end{array} \arrow[r, "\cong"] \arrow[d, "\text{split}", "\cong"'] & \begin{array}{c} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n-r))[2r], \mathbb{R})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-1][-2r] \end{array} \arrow[d, "\text{split}", "\cong"'] \\
R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{R}(n)) \arrow[r, "\cong"] & R\Gamma_{W,c}(X, \mathbb{R}(n-r))[-2r]
\end{tikzcd}$$

Here the first square is commutative due to the compatibility of the regulator with affine bundles (Lemma 6.2), and the second square commutes because the quasi-isomorphism (6.12) gives compatible splittings (see again Figure 4 on p. 33). Taking the determinants, we obtain the desired commutative diagram (6.13). □

$$\begin{tikzcd}
 & & & & R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{Z}(n)) \arrow[dl] \arrow[dd, dashed, "\cong"] & \\
R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \arrow[r, "\alpha_{\mathbb{A}^r_X,n}"] \arrow[dd, "\cong"', "(p^*)^\vee"] & R\Gamma_c(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}(n)) \arrow[rr] \arrow[dr, "u_\infty"] \arrow[dd, "\cong"', "p_*"] & & R\Gamma_{fg}(\mathbb{A}^r_X, \mathbb{Z}(n)) \arrow[rr] \arrow[dl, "i_\infty"'] \arrow[dd, dashed, "\cong"] & & \cdots[-1] \arrow[dd, "\cong"] \\
 & & R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{Z}(n)) \arrow[dd, "\cong"', "p_*"] & & R\Gamma_{W,c}(X, \mathbb{Z}(n-r))[-2r] \arrow[dl] & \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r], \mathbb{Q}[-2]) \arrow[r, "\alpha_{X,n-r}[-2r]"] & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n-r))[-2r] \arrow[rr] \arrow[dr, "u_\infty"] & & R\Gamma_{fg}(X, \mathbb{Z}(n-r))[-2r] \arrow[rr] \arrow[dl, "i_\infty"'] & & \cdots[-1] \\
 & & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{Z}(n-r))[-2r] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{R}(n)) \arrow[dl] \arrow[dd, dashed, "\cong"] & \\
R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}[-2]) \arrow[r] \arrow[dd, "\cong"', "(p^*)^\vee"] & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & R\Gamma_{fg}(\mathbb{A}^r_X, \mathbb{R}(n)) \arrow[rr] \arrow[dl, "0"'] \arrow[dd, dashed, "\cong"] & & \cdots[-1] \arrow[dd, "\cong"] \\
 & & R\Gamma_c(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), \mathbb{R}(n)) \arrow[dd, "\cong"', "p_*"] & & R\Gamma_{W,c}(X, \mathbb{R}(n-r))[-2r] \arrow[dl] & \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r], \mathbb{R}[-2]) \arrow[r] & 0 \arrow[rr] \arrow[dr] & & R\Gamma_{fg}(X, \mathbb{R}(n-r))[-2r] \arrow[rr] \arrow[dl, "0"'] & & \cdots[-1] \\
 & & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n-r))[-2r] & & &
\end{tikzcd}$$

Figure 4: Isomorphism $R\Gamma_{W,c}(\mathbb{A}^r_X, \mathbb{Z}(n)) \cong R\Gamma_{W,c}(X, \mathbb{Z}(n-r))[-2r]$ and its splitting after tensoring with $\mathbb{R}$.

Theorem 6.8. *For an arithmetic scheme X and $n < 0$, assume $\mathbf{L}^c(X_{ét}, n)$, $\mathbf{B}(X, n)$, and the meromorphic continuation of $\zeta(X, s)$ around $s = n$.*

1) *If $X = \coprod_{1 \le i \le r} X_i$ is a finite disjoint union of arithmetic schemes, then*
$$\mathbf{C}(X, n) \iff \mathbf{C}(X_i, n) \text{ for all } i.$$

2) *For a closed-open decomposition $Z \not\hookrightarrow X \hookleftarrow U$, if two of three conjectures*
$$\mathbf{C}(X, n), \quad \mathbf{C}(Z, n), \quad \mathbf{C}(U, n)$$
are true, then the third is also true.

3) *For any $r \ge 0$, one has*
$$\mathbf{C}(\mathbb{A}^r_X, n) \iff \mathbf{C}(X, n - r).$$

Proof. Follows from Lemmas 6.4, 6.5, 6.7, together with the corresponding identities for the zeta functions (6.1), (6.2), (6.3). ☐

The following is a special case of compatibility with closed-open decompositions.

Lemma 6.9. *For an arithmetic scheme X and $n < 0$, the conjectures*
$$\mathbf{L}^c(X_{ét}, n), \ \mathbf{B}(X, n), \ \mathbf{VO}(X, n), \ \mathbf{C}(X, n)$$
are equivalent to
$$\mathbf{L}^c(X_{red,ét}, n), \ \mathbf{B}(X_{red}, n), \ \mathbf{VO}(X_{red}, n), \ \mathbf{C}(X_{red}, n)$$
respectively.

Proof. Apply Lemma 6.1, Lemma 6.2, Proposition 6.3, and Theorem 6.8 to the canonical closed embedding $X_{red} \hookrightarrow X$. ☐

The above lemma can be proved directly, by going through the construction of Weil-étale cohomology in [2] and the statements of the conjectures. In particular,
$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong R\Gamma_{W,c}(X_{red}, \mathbb{Z}(n)).$$
It is important to note that the cycle complexes do not distinguish X from X_{red}, and neither does the zeta function: $\zeta(X, s) = \zeta(X_{red}, s)$.

Remark 6.10. If $X/\mathbb{F}_q$ is a variety over a finite field, then the proof of Theorem 6.8 simplifies drastically: we can work with the formula (5.1) and the following properties of motivic cohomology:

1) $R\Gamma(\coprod_i X_{i,ét}, \mathbb{Z}^c(n)) \cong \bigoplus_i R\Gamma(X_{i,ét}, \mathbb{Z}^c(n))$;

2) triangles associated to closed-open decompositions
$$R\Gamma(Z_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(U_{ét}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{ét}, \mathbb{Z}^c(n))[1]$$

3) homotopy invariance $R\Gamma(X_{ét}, \mathbb{Z}^c(n - r))[2r] \cong R\Gamma(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n))$.

In this case, no regulators are involved, so we do not need the technical lemmas 6.4, 6.5, 6.7.

If we consider the projective space $\mathbb{P}^r_X = \mathbb{P}^r_{\mathbb{Z}} \times X$, we have a formula for the zeta function

$$\zeta(\mathbb{P}^r_X, s) = \prod_{0 \le i \le r} \zeta(X, s - i). \tag{6.14}$$

Our special value conjecture satisfies the corresponding compatibility.

Corollary 6.11 (Projective bundles). *Let X be an arithmetic scheme, $n < 0$, and $r \ge 0$. For all $0 \le i \le r$ assume Conjectures $\mathbf{L}^c(X_{ét}, n-i)$, $\mathbf{B}(X, n-i)$, and the meromorphic continuation of $\zeta(X,s)$ around $s = n - i$. Then*

$$\mathbf{C}(X, n-i) \text{ for } 0 \le i \le r \Longrightarrow \mathbf{C}(\mathbb{P}^r_X, n).$$

Proof. Applied to the closed-open decomposition $\mathbb{P}^{r-1}_X \not\hookrightarrow \mathbb{P}^r_X \hookleftarrow \mathbb{A}^r_X$, Theorem 6.8 gives

$$\mathbf{C}(X, n-r) \text{ and } \mathbf{C}(\mathbb{P}^{r-1}_X, n) \Longrightarrow \mathbf{C}(\mathbb{A}^r_X, n) \text{ and } \mathbf{C}(\mathbb{P}^{r-1}_X, n) \Longrightarrow \mathbf{C}(\mathbb{P}^r_X, n).$$

The assertion follows by induction on r. (The same inductive argument proves the identity (6.14) from (6.3).) ☐

7 Unconditional results

Now we apply Theorem 6.8 to prove the main theorem stated in the introduction: the validity of $\mathbf{VO}(X,n)$ and $\mathbf{C}(X,n)$ for all $n < 0$ for cellular schemes over certain one-dimensional bases. In fact, we will construct an even larger class of schemes $\mathcal{C}(\mathbb{Z})$ whose elements satisfy the conjectures. This approach is motivated by [39, §5].

Definition 7.1. Let $\mathcal{C}(\mathbb{Z})$ be the full subcategory of the category of arithmetic schemes generated by the following objects:

- the empty scheme $\varnothing$,
- $\operatorname{Spec} \mathbb{F}_q$ for each finite field,
- $\operatorname{Spec} \mathcal{O}_F$ for an abelian number field $F/\mathbb{Q}$,
- curves over finite fields $C/\mathbb{F}_q$,

and the following operations.

$\mathcal{C}0$) X is in $\mathcal{C}(\mathbb{Z})$ if and only if X_{red} is in $\mathcal{C}(\mathbb{Z})$.

$\mathcal{C}1$) A finite disjoint union $\coprod_{1 \le i \le r} X_i$ is in $\mathcal{C}(\mathbb{Z})$ if and only if each X_i is in $\mathcal{C}(\mathbb{Z})$.

$\mathcal{C}2$) Let $Z \not\hookrightarrow X \hookleftarrow U$ be a closed-open decomposition such that $Z_{red,\mathbb{C}}$, $X_{red,\mathbb{C}}$, $U_{red,\mathbb{C}}$ are smooth and quasi-projective. If two of the three schemes Z, X, U lie in $\mathcal{C}(\mathbb{Z})$, then the third also lies in $\mathcal{C}(\mathbb{Z})$.

$\mathcal{C}3$) If X lies in $\mathcal{C}(\mathbb{Z})$, then the affine space $\mathbb{A}^r_X$ for each $r \ge 0$ also lies in $\mathcal{C}(\mathbb{Z})$.

Recall that the condition that $X_{red,\mathbb{C}}$ is smooth and quasi-projective is necessary to ensure that the regulator morphism exists (see Remark 2.7).

Proposition 7.2. *Conjectures* $\mathbf{VO}(X,n)$ *and* $\mathbf{C}(X,n)$ *hold for any* $X \in \mathcal{C}(\mathbb{Z})$ *and* $n < 0$.

Proof. Finite fields satisfy $\mathbf{C}(X,n)$ by Example 5.5.

If $X = \operatorname{Spec} \mathcal{O}_F$ for a number field $F/\mathbb{Q}$, then Conjecture $\mathbf{C}(X,n)$ is equivalent to the conjecture of Flach and Morin [11, Conjecture 5.12], which holds unconditionally for abelian $F/\mathbb{Q}$, via reduction to the Tamagawa number conjecture; see [11, §5.8.3], in particular [ibid., Proposition 5.35]. The condition $\mathbf{VO}(X,n)$ is also true in this case (see Example 3.7).

If $X = C/\mathbb{F}_q$ is a curve over a finite field, then $\mathbf{C}(X,n)$ holds thanks to Theorem 5.3. Conjecture $\mathbf{L}^c(X_{ét},n)$ is known for curves and essentially goes back to Soulé; see, for example, [19, Proposition 4.3].

Finally, the fact that the Conjectures $\mathbf{L}^c(X_{ét},n)$, $\mathbf{B}(X,n)$, $\mathbf{VO}(X,n)$, $\mathbf{C}(X,n)$ are closed under the operations $\mathcal{C}0$)–$\mathcal{C}3$) is Lemma 6.9, Lemma 6.1, Lemma 6.2, Proposition 6.3, and Theorem 6.8 respectively. ☐

Lemma 7.3. *Any zero-dimensional arithmetic scheme* X *is in* $\mathcal{C}(\mathbb{Z})$.

Proof. Since X is a Noetherian scheme of dimension 0, it is a finite disjoint union of $\operatorname{Spec} A_i$ for some Artinian local rings A_i. Thanks to $\mathcal{C}1$), we can assume that $X = \operatorname{Spec} A$, and thanks to $\mathcal{C}0$), we can assume that X is reduced. But then $A = k$ is a field. Since X is a scheme of finite type over $\operatorname{Spec} \mathbb{Z}$, we conclude that $X = \operatorname{Spec} \mathbb{F}_q \in \mathcal{C}(\mathbb{Z})$. ☐

Proposition 7.4. *Let* B *be a one-dimensional arithmetic scheme. Suppose that each of the generic points* $\eta \in B$ *satisfies one of the following properties:*

a) $\operatorname{char} \kappa(\eta) = p > 0$;

b) $\operatorname{char} \kappa(\eta) = 0$, *and* $\kappa(\eta)/\mathbb{Q}$ *is an abelian number field.*

Then $B \in \mathcal{C}(\mathbb{Z})$.

Proof. We verify that such a scheme can be obtained from $\operatorname{Spec} \mathcal{O}_F$ for an abelian number field $F/\mathbb{Q}$, or a curve over a finite field $C/\mathbb{F}_q$, using the operations $\mathcal{C}0$), $\mathcal{C}1$), $\mathcal{C}2$) which appear in the definition of $\mathcal{C}(\mathbb{Z})$.

Thanks to $\mathcal{C}0$), we can assume that B is reduced. Consider the normalization $\nu\colon B' \to B$. This is a birational morphism, so there exist open dense subsets $U' \subseteq B'$ and $U \subseteq B$ such that $\nu|_{U'} : U' \xrightarrow{\cong} U$. Now $B \setminus U$ is zero-dimensional, and therefore $B \setminus U \in \mathcal{C}(\mathbb{Z})$ by the previous lemma. Thanks to $\mathcal{C}2$), it suffices to check that $U' \in \mathcal{C}(\mathbb{Z})$, and this would imply $B \in \mathcal{C}(\mathbb{Z})$.

Now U' is a finite disjoint union of normal integral schemes, so according to $\mathcal{C}1$) we can assume that U' is integral. Consider the generic point $\eta \in U'$ and the residue field $F = \kappa(\eta)$. There are two cases to consider.

a) If $\operatorname{char} F = p > 0$, then U' is a curve over a finite field, so it lies in $\mathcal{C}(\mathbb{Z})$.

b) If $\operatorname{char} F = 0$, then by our assumptions, $F/\mathbb{Q}$ is an abelian number field.

   We note that if $V' \subseteq U'$ is an affine open neighborhood of η, then $U' \setminus V' \in \mathcal{C}(\mathbb{Z})$ by the previous lemma. Therefore, we can assume without loss of generality that U' is affine.

   We have $U' = \operatorname{Spec} \mathcal{O}$, where $\mathcal{O}$ is a finitely generated integrally closed domain. This means that $\mathcal{O}_F \subseteq \mathcal{O} = \mathcal{O}_{F,S}$ for a finite set of places S. Now $U' = \operatorname{Spec} \mathcal{O}_F \setminus S$, and $S \in \mathcal{C}(\mathbb{Z})$, so everything reduces to the case of $U' = \operatorname{Spec} \mathcal{O}_F$, which is in $\mathcal{C}(\mathbb{Z})$. ☐

Remark 7.5. Schemes like the above were considered by Jordan and Poonen in [23], where the authors write down a special value formula for $s = 1$ that generalizes the classical class number formula. Namely, they consider the case where B is reduced and affine, but without requiring $\kappa(\eta)/\mathbb{Q}$ to be abelian.

Example 7.6. If $B = \operatorname{Spec} \mathcal{O}$ for a nonmaximal order $\mathcal{O} \subset \mathcal{O}_F$, where $F/\mathbb{Q}$ is an abelian number field, then our formalism gives a cohomological interpretation of the special values of $\zeta_{\mathcal{O}}(s)$ at $s = n < 0$. This already seems to be a new result.

Definition 7.7. Let $X \to B$ be a B-scheme. We say that X is **B-cellular** if it admits a filtration by closed subschemes

$$X = Z_N \supseteq Z_{N-1} \supseteq \cdots \supseteq Z_0 \supseteq Z_{-1} = \varnothing \tag{7.1}$$

such that $Z_i \setminus Z_{i-1} \cong \coprod_j \mathbb{A}_B^{r_{i_j}}$ is a finite union of affine B-spaces.

For example, projective spaces $\mathbb{P}^r_B$ and, in general, Grassmannians $\operatorname{Gr}(k,\ell)_B$ are cellular. Many interesting examples of cellular schemes arise from actions of algebraic groups on varieties and the Białynicki-Birula theorem; see [48] and [6].

Proposition 7.8. *Let X be a B-cellular arithmetic scheme, where $B \in \mathcal{C}(\mathbb{Z})$, and $X_{red,\mathbb{C}}$ is smooth and quasi-projective. Then $X \in \mathcal{C}(\mathbb{Z})$.*

Proof. Considering the corresponding cellular decomposition (7.1), we pass to open complements $U_i = X \setminus Z_i$ to obtain a filtration

$$X = U_{-1} \supseteq U_1 \supseteq \cdots \supseteq U_{N-1} \supseteq U_N = \varnothing,$$

where $U_{i,\mathbb{C}}$ are smooth and quasi-projective, being *open* subvarieties in $X_{\mathbb{C}}$. Now we have closed-open decompositions $\coprod_j \mathbb{A}_B^{r_{i,j}} \not\hookrightarrow U_i \hookleftarrow U_{i+1}$, and the claim follows by induction on the length of the cellular decomposition, using operations $\mathcal{C}1$)–$\mathcal{C}3$). ☐

As a corollary of the above, we obtain the following result, stated in the introduction.

Theorem 7.9. *Let B be a one-dimensional arithmetic scheme satisfying the assumptions of Proposition 7.4. If X is a B-cellular arithmetic scheme with smooth and quasi-projective fiber $X_{red,\mathbb{C}}$, then Conjectures $\mathbf{VO}(X,n)$ and $\mathbf{C}(X,n)$ hold unconditionally for any $n < 0$.*

Proof. Follows from propositions 7.2, 7.4, 7.8. ☐

A Determinants of complexes

Here we give a brief overview of the determinants of complexes. The original construction goes back to Knudsen and Mumford [31], and useful expositions can be found in [20, Appendix A] and [25, §2.1].

For our purposes, let R be an integral domain.

Definition A.1. Denote by $\mathcal{P}_{is}(R)$ the category of **graded invertible R-modules**. It has as objects (L, r), where L is an invertible R-module (i.e. projective of rank 1) and $r \in \mathbb{Z}$. The morphisms in this category are given by

$$\mathrm{Hom}_{\mathcal{P}_{is}(R)}((L,r),(M,s)) = \begin{cases} \mathrm{Isom}_R(L,M), & r = s, \\ \varnothing, & r \neq s. \end{cases}$$

This category is equipped with tensor products

$$(L,r) \otimes_R (M,s) = (L \otimes_R M, r+s)$$

with (graded) commutativity isomorphisms

$$(L,r) \otimes_R (M,s) \xrightarrow{\cong} (M,s) \otimes_R (L,r), \quad \ell \otimes m \mapsto (-1)^{rs}\, m \otimes \ell.$$

The unit object with respect to this product is $(R,0)$, and for each $(L,r) \in \mathcal{P}_{is}(R)$ the inverse is given by $(L^{-1}, -r)$ where $L^{-1} = \underline{\mathrm{Hom}}_R(L,R)$. The canonical evaluation morphism $L \otimes_R \underline{\mathrm{Hom}}_R(L,R) \to R$ induces an isomorphism

$$(L,r) \otimes_R (L^{-1}, -r) \cong (R,0).$$

Definition A.2. Denote by $\mathcal{C}_{is}(R)$ the category whose objects are finitely generated projective R-modules and whose morphisms are isomorphisms. For $A \in \mathcal{C}_{is}(R)$ we define the corresponding determinant by

$$\det_R(A) = \Big(\bigwedge_R^{\mathrm{rk}_R A} A, \mathrm{rk}_R A \Big) \in \mathcal{P}_{is}(R). \tag{A.1}$$

Here $\mathrm{rk}_R A$ is the rank of A, so that the top exterior power $\bigwedge_R^{\mathrm{rk}_R A} A$ is an invertible R-module.

This yields a functor $\det_R \colon \mathcal{C}_{is}(R) \to \mathcal{P}_{is}(R)$. For $(L,r) \in \mathcal{P}_{is}(R)$ we usually forget about r and treat the determinant as an invertible R-module.

The main result of [31, Chapter I] is that this construction can be generalized to complexes and morphisms in the derived category.

Definition A.3. Let $\mathbf{D}(R)$ be the derived category of the category of R-modules. Recall that a complex $A^\bullet$ is **perfect** if it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. We denote by $\mathcal{P}\mathit{arf}_{is}(R)$ the subcategory of $\mathbf{D}(R)$ whose objects consist of perfect complexes, and whose morphisms are quasi-isomorphisms of complexes.

Theorem A.4 (Knudsen–Mumford). *The determinant* (A.1) *extends to perfect complexes of R-modules as a functor*

$$\det_R \colon \mathcal{P}\mathit{arf}_{is}(R) \to \mathcal{P}_{is}(R),$$

satisfying the following properties.

- $\det_R(0) = (R, 0)$.
- *For a distinguished triangle of complexes in* $\mathcal{P}arf_{is}(R)$

  $$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

  there is a canonical isomorphism

  $$i_R(u, v, w)\colon \det_R A^\bullet \otimes_R \det_R C^\bullet \xrightarrow{\cong} \det_R B^\bullet.$$

- *In particular, there exist canonical isomorphisms*

  $$\det_R(A^\bullet \oplus B^\bullet) \cong \det_R(A^\bullet) \otimes_R \det_R(B^\bullet).$$

- *For the triangles*

  $$\begin{array}{l} A^\bullet \xrightarrow{id} A^\bullet \longrightarrow 0^\bullet \longrightarrow A^\bullet[1] \\ 0^\bullet \longrightarrow A^\bullet \xrightarrow{id} A^\bullet \longrightarrow 0^\bullet[1] \end{array}$$

  the isomorphism i_R *comes from the canonical isomorphism* $\det_R A^\bullet \otimes_R (R, 0) \cong \det_R A^\bullet$.

- *For an isomorphism of distinguished triangles*

  $$\begin{tikzcd} A^\bullet \arrow[r, "u"] \arrow[d, "f", "\cong"'] & B^\bullet \arrow[r, "v"] \arrow[d, "g", "\cong"'] & C^\bullet \arrow[r, "w"] \arrow[d, "h", "\cong"'] & A^\bullet[1] \arrow[d, "f[1]", "\cong"'] \\ A'^\bullet \arrow[r, "u'"] & B'^\bullet \arrow[r, "v'"] & C'^\bullet \arrow[r, "w'"] & A'^\bullet[1] \end{tikzcd}$$

  the diagram

  $$\begin{tikzcd} \det_R A^\bullet \otimes_R \det_R C^\bullet \arrow[r, "i_R(u,v,w)", "\cong"'] \arrow[d, "\det_R(f)\otimes\det_R(h)", "\cong"'] & \det_R B^\bullet \arrow[d, "\det_R(g)", "\cong"'] \\ \det_R A'^\bullet \otimes_R \det_R C'^\bullet \arrow[r, "i_R(u',v',w')", "\cong"'] & \det_R B'^\bullet \end{tikzcd}$$

  is commutative.

- *The determinant is compatible with base change: given a ring homomorphism* $R \to S$*, there is a natural isomorphism*

  $$\det_S(A^\bullet \otimes^{\mathbf{L}}_R S) \xrightarrow{\cong} (\det_R A^\bullet) \otimes_R S.$$

  Moreover, this isomorphism is compatible with i_R *and* i_S.

- *If* $A^\bullet$ *is a bounded complex where each object* A^i *is perfect (i.e. admits a finite length resolution by finitely generated projective* R*-modules), then*

  $$\det_R A^\bullet \cong \bigotimes_{i \in \mathbb{Z}} (\det_R A^i)^{(-1)^i}.$$

  If each A^i *is already a finitely generated projective* R*-module, then* $\det_R A^i$ *in the above formula is given by* (A.1).

- *If the cohomology modules $H^i(A^\bullet)$ are perfect, then*

$$\det_R A^\bullet \cong \bigotimes_{i\in\mathbb{Z}} (\det_R H^i(A^\bullet))^{(-1)^i}. \tag{A.2}$$

We refer the reader to [31] for the actual construction and proofs.

A particularly simple case of interest is when $R = \mathbb{Z}$ and all cohomology groups $H^i(A^\bullet)$ are finite.

Lemma A.5.

1) *Let A be a finite abelian group. Then*

$$(\det_{\mathbb{Z}} A) \subset (\det_{\mathbb{Z}} A) \otimes \mathbb{Q} \cong \det_{\mathbb{Q}}(A \otimes \mathbb{Q}) = \det_{\mathbb{Q}}(0) \cong \mathbb{Q}$$

corresponds to the fractional ideal $\frac{1}{\#A}\mathbb{Z} \subset \mathbb{Q}$.

2) *In general, let $A^\bullet$ be a perfect complex of abelian groups such that the cohomology groups $H^i(A^\bullet)$ are all finite. Then $\det_{\mathbb{Z}} A^\bullet$ corresponds to the fractional ideal $\frac{1}{m}\mathbb{Z} \subset \mathbb{Q}$, where*

$$m = \prod_{i\in\mathbb{Z}} |H^i(A^\bullet)|^{(-1)^i}.$$

Proof. Since $\det_{\mathbb{Z}}(A \oplus B) \cong \det_{\mathbb{Z}} A \otimes \det_{\mathbb{Z}} B$, in part 1) it suffices to consider the case of a cyclic group $A = \mathbb{Z}/m\mathbb{Z}$. Using the resolution

$$\mathbb{Z}/m\mathbb{Z}[0] \cong \Bigl[\underset{\text{deg.}\,-1}{m\mathbb{Z}} \hookrightarrow \underset{\text{deg.}\,0}{\mathbb{Z}}\Bigr],$$

we calculate

$$\det_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z} \otimes (m\mathbb{Z})^{-1} \cong (m\mathbb{Z})^{-1},$$

which corresponds to $\frac{1}{m}\mathbb{Z}$ in $\mathbb{Q}$. Part 2) follows directly from 1) and (A.2). ☐

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