Zeta-values from Euler to Weil-étale cohomology

Alexey Beshenov
Université de Bordeaux / Universiteit Leiden

Advised by Baptiste Morin and Bas Edixhoven

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Outline

- XVIII century mathematics: $\sum_{n \geq 1} \frac{1}{n^{2k}}$.
- XIX century mathematics: $\zeta(s)$ and $\zeta_F(s)$.
- XX century mathematics: $\zeta_X(s)$.
- Algebraic K-theory.
- Motivic cohomology.
- Weil-étale cohomology.

Riemann zeta function before Riemann

- **Pietro Mengoli**, 1644, the "Basel problem": $\sum_{n\geq 1} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots =$?????
- **Euler** (*"De summis serierum reciprocarum"*, 1740): $\frac{\pi^2}{6}$.
- In general (ibid.), $\sum_{n\geq 1} \frac{1}{n^{2k}} = (-1)^{k+1} B_{2k} \frac{2^{2k-1}}{(2k)!} \pi^{2k}$.
- Bernoulli numbers: $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, $B_7 = 0$, $B_8 = -\frac{1}{30}$, $B_9 = 0$, $B_{10} = \frac{5}{66}$, ... (**Jacob Bernoulli**, *"Ars Conjectandi"*, 1713).
- Faulhaber's formula: $\sum_{1\leq i\leq n} i^k = \frac{1}{k+1} \sum_{0\leq i\leq k} \binom{k+1}{i} B_i n^{k+1-i}$ (**Johann Faulhaber**, *"Academia Algebræ"*, 1631; Bernoulli, 1713).

Riemann zeta function

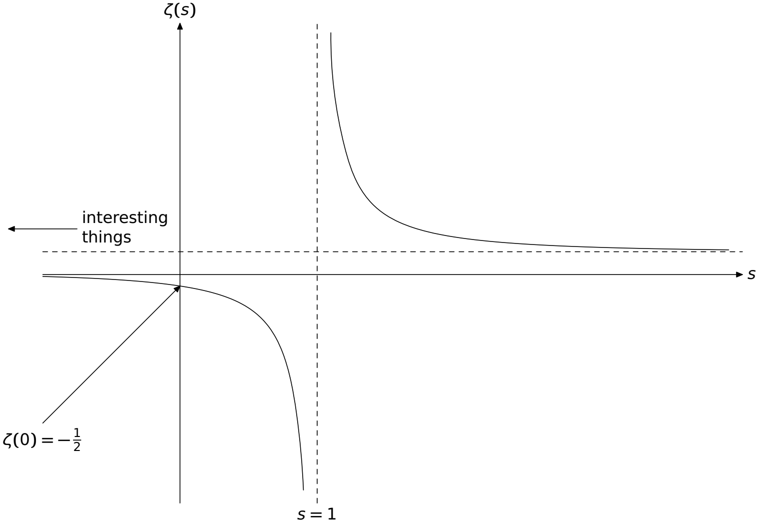
Riemann, *"Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"* (1859):

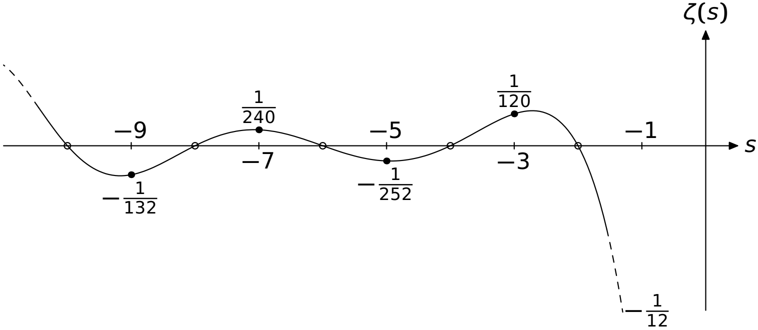
- $\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} \qquad (\operatorname{Re} s > 1)$.
- **Euler** (*"Variæ observationes circa series infinitas"*, 1744): $= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$.
- Meromorphic continuation to $\mathbb{C}$ with one simple pole at $s = 1$.
- Functional equation

$$\zeta(1-s) = 2\,(2\pi)^{-s}\,\Gamma(s)\,\cos\left(\frac{\pi s}{2}\right)\,\zeta(s).$$

- (Trivial) simple zeros at $s = -2, -4, -6, \ldots$
- Euler's calculation $\zeta(2k) = (-1)^{k+1}\,B_{2k}\,\frac{2^{2k-1}}{(2k)!}\,\pi^{2k}$ becomes

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \text{ for } n = 1, 2, 3, 4, \ldots$$


$\zeta(s)$
interesting things
s
$\zeta(0) = -\frac{1}{2}$
$s = 1$


$\zeta(s)$
s
-9
-7
-5
-3
-1
$\frac{1}{240}$
$\frac{1}{120}$
$-\frac{1}{132}$
$-\frac{1}{252}$
$-\frac{1}{12}$

Riemann zeta function at positive odd integers

- **Roger Apéry**, 1977: $\zeta(3) = 1.2020569...$ is irrational.
- **Tanguy Rivoal**, 2000: infinitely many irrationals $\zeta(2k+1)$.
- **Wadim Zudilin**, 2001: at least one irrational among $\zeta(5),\ \zeta(7),\ \zeta(9),\ \zeta(11)$ (which one?)
- Conjecture: $\zeta(2k+1)$ are transcendental, algebraically independent.

Dedekind zeta function: definition

- $F/\mathbb{Q}$ — number field, $\mathcal{O}_F :=$ the ring of integers.
- **Dedekind**, appendix to Dirichlet's "Vorlesungen über Zahlentheorie" (1863):

$$\zeta_F(s) := \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ \mathfrak{a} \neq 0}} \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{a})^s} = \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_F \\ \text{prime}}} \frac{1}{1 - N_{F/\mathbb{Q}}(\mathfrak{p})^{-s}}. \qquad (\operatorname{Re} s > 1)$$

- Note: $\zeta_{\mathbb{Q}}(s) = \zeta(s)$.

Dedekind zeta function: functional equation

- **Hecke**, *"Über die Zetafunktion beliebiger algebraischer Zahlkörper"*, 1917: meromorphic continuation with simple pole at $s = 1$; functional equation

$$\zeta_F(1-s) = |\Delta_F|^{s-1/2} \left(\cos \tfrac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \tfrac{\pi s}{2}\right)^{r_2} \left(2\,(2\pi)^{-s}\,\Gamma(s)\right)^d \zeta_F(s),$$

  where
  $r_1 :=$ real places, $r_2 :=$ conjugate pairs of complex places;
  $d := [F : \mathbb{Q}] = r_1 + 2\,r_2$ and $\Delta_F :=$ discriminant.

- (Trivial) zeros:

s:	0	-1	-2	-3	-4	-5	$\cdots$
order:	$r_1 + r_2 - 1$	r_2	$r_1 + r_2$	r_2	$r_1 + r_2$	r_2	$\cdots$

Class number formula

Dirichlet, *"Recherches sur diverses applications de l'analyse infinitésimale à la théorie des nombres"* (1839);
* **Gauss**, *"Disquisitiones Arithmeticæ"* (1801):

- Pole at $s = 1$:

$$\lim_{s \to 1} (s-1)\,\zeta_F(s) = \frac{2^{r_1}\,(2\pi)^{r_2}\,\#\,\mathrm{Cl}(F)}{\#\mu_F \cdot \sqrt{|\Delta_F|}}\,R_F,$$

  where $\mathrm{Cl}(F)$ — class group; $\mu_F \subset \mathcal{O}_F^\times$ — roots of unity; R_F — Dirichlet regulator.

- Zero at $s = 0$:

$$\lim_{s \to 0} s^{-(r_1+r_2-1)}\,\zeta_F(s) = -\frac{\#\,\mathrm{Cl}(F)}{\#\mu_F}\,R_F.$$

Values of the Dedekind zeta function

- If F is totally real ($r_2 = 0$), then $\zeta_F(-n) \neq 0$ for odd n.
- "Siegel–Klingen theorem", 1961: $\zeta_F(-n) \in \mathbb{Q}$.
- **Günter Harder**, "A Gauss–Bonnet formula for discrete arithmetically defined groups", 1971:

$$\chi(\mathrm{Sp}_{2n}(\mathcal{O}_F)) = \frac{1}{2^{n\,(d-n)}} \prod_{1 \le i \le n} \zeta_F(1-2n).$$

- Example: $F = \mathbb{Q}$, $n = 1$, $\mathrm{Sp}_2 = \mathrm{SL}_2$,

$$\chi(\mathrm{SL}_2(\mathbb{Z})) = -\frac{1}{12} = -\frac{B_2}{2} = \zeta(-1)$$

("orbifold Euler characteristic" of $\mathcal{H}/\,\mathrm{SL}_2(\mathbb{Z})$).

Zeta function of a scheme

- $X \to \operatorname{Spec}\mathbb{Z}$ — arithmetic scheme (separated, of finite type).
- $\zeta_X(s) := \prod_{x \in X_0} \frac{1}{1-N(x)^{-s}}$. $\qquad$ ($\operatorname{Re} s > \dim X$).
  $X_0 :=$ closed points; $N(x) := \#$residue field at x.
- Note: $\zeta_{\operatorname{Spec}\mathbb{Z}}(s) = \zeta(s)$ and $\zeta_{\operatorname{Spec}\mathcal{O}_F}(s) = \zeta_F(s)$.
- Conjecture (!): meromorphic continuation and a functional equation $\zeta_X(s) \leftrightarrow \zeta_X(\dim X - s)$.
- Special values may be studied via
  K-theory $K_n(X)$ or motivic cohomology $H^i(X, \mathbb{Z}(n))$.

Algebraic K-theory

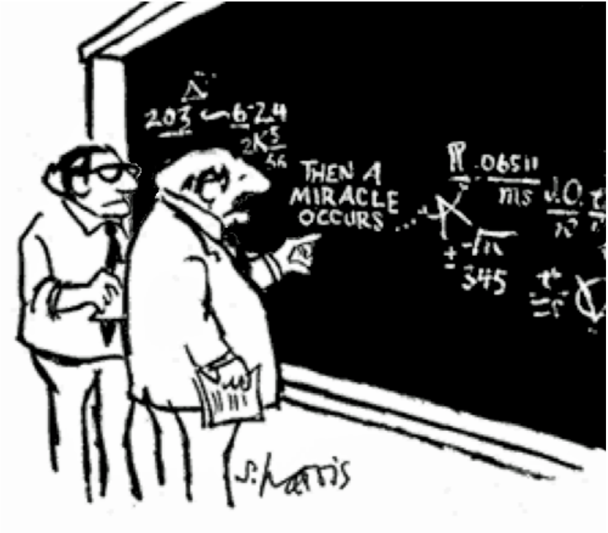
- Input: an "exact category" $\mathcal{C}$.
  Examples: $\mathbf{VB}(X)$ and R-$\mathbf{Proj}_{fg} \simeq \mathbf{VB}(\operatorname{Spec} R)$.
- **Grothendieck**, 1957
  (work on [Grothendieck–Hirzebruch]–Riemann–Roch):

$$K_0(\mathcal{C}) := \frac{\mathbb{Z}\,\langle \text{isomorphism classes of objects of } \mathcal{C}\rangle}{[B] = [A] + [C] \text{ for each s.e.s. } 0 \to A \to B \to C \to 0}.$$

- **Quillen**, 1973:

$$K_0(\mathcal{C}) \cong \pi_1(BQ\mathcal{C}, 0),$$
$$K_n(\mathcal{C}) := \pi_{n+1}(BQ\mathcal{C}, 0);$$

Q — Quillen's "Q-construction",
B — geometric realization of the nerve.



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

© Sidney Harris

Some of the few known calculations

- **Quillen**, 1972:

$$\begin{aligned} K_0(\mathbb{F}_q) &\cong \mathbb{Z}, \\ K_{2n}(\mathbb{F}_q) &= 0, \\ K_{2n-1}(\mathbb{F}_q) &\cong \mathbb{Z}/(q^n - 1)\mathbb{Z}. \end{aligned}$$

- Note: $\#K_{2n-1}(\mathbb{F}_q) = -\zeta_{\mathbb{F}_q}(-n)^{-1}$.
- **Quillen**, 1973: $K_n(\mathcal{O}_F)$ are finitely generated.
- **Armand Borel**, 1974:

$$\operatorname{rk} K_n(\mathcal{O}_F) = \begin{cases} 0, & n = 2k, \\ r_1 + r_2, & n = 4k+1, \\ r_2, & n = 4k-1. \end{cases} \qquad (k > 0)$$

- Note: $\operatorname{rk} K_{2n+1}(\mathcal{O}_F) =$ order of zero of $\zeta_F(s)$ at $s = -n$.

Torsion in the K-theory of $\mathbb{Z}$

- **Milnor**, 1971: $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
- **Lee**, **Szczarba**, 1976: $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$.
- **Rognes**, 2000: $K_4(\mathbb{Z}) = 0$.
- **Elbaz-Vincent**, **Gangl**, **Soulé**, 2002: $K_5(\mathbb{Z}) \cong \mathbb{Z}$.
- Using the Bloch–Kato conjecture (**Voevodsky**, **Rost**, ...):

n:	2	3	4	5
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/48\mathbb{Z}$	0	$\mathbb{Z}$
n:	6	7	8	9
$K_n(\mathbb{Z})$:	0	$\mathbb{Z}/240\mathbb{Z}$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
n:	10	11	12	13
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/1008\mathbb{Z}$	(0?)	$\mathbb{Z}$
n:	14	15	16	17
$K_n(\mathbb{Z})$:	0	$\mathbb{Z}/480\mathbb{Z}$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
n:	18	19	20	21
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/528\mathbb{Z}$	(0?)	$\mathbb{Z}$
n:	22	23	24	25
$K_n(\mathbb{Z})$:	$\mathbb{Z}/691\mathbb{Z}$	$\mathbb{Z}/65\,520\mathbb{Z}$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Lichtenbaum's conjecture

- Easy: $K_0(\mathcal{O}_F) \cong \mathrm{Cl}(F) \oplus \mathbb{Z}$.
- Not-so-easy (**Bass**, **Milnor**, **Serre**, 1967): $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times$.
- Dirichlet's unit theorem: $\mathcal{O}_F^\times \cong \mathbb{Z}^{r_1+r_2-1} \oplus \mu_F$.
- Class number formula:

$$\lim_{s\to 0} s^{-(r_1+r_2-1)}\,\zeta_F(s) = -\frac{\#K_0(\mathcal{O}_F)_{tors}}{\#K_1(\mathcal{O}_F)_{tors}}\,R_F.$$

- **Lichtenbaum**, 1973:

$$\lim_{s\to n}(n-s)^{-\mu_n}\,\zeta_F(-s) = \pm 2^{?}\,\frac{\#K_{2n}(\mathcal{O}_F)}{\#K_{2n+1}(\mathcal{O}_F)_{tors}}\,R_{F,n}.$$

  $R_{F,n}$ — "higher regulators" (Borel, Beĭlinson).
- Example: $\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}$, $\frac{\#K_2(\mathbb{Z})}{\#K_3(\mathbb{Z})} = \frac{\#\mathbb{Z}/2}{\#\mathbb{Z}/48} = \frac{1}{24}$.
- Example: $\zeta(-11) = -\frac{B_{12}}{12} = \frac{691}{12\cdot 2730} = \frac{691}{2^3\cdot 3^2\cdot 5\cdot 7\cdot 13}$,
  $\frac{\#K_{22}(\mathbb{Z})}{\#K_{23}(\mathbb{Z})} = \frac{691}{65520} = \frac{691}{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}$.

Motivic cohomology

- **Bloch**, 1986: complexes of Zariski / étale sheaves $\mathbb{Z}(n)$.
- $H^i(X_{\text{Zar}}, \mathbb{Z}(n))$, $H^i(X_{\text{ét}}, \mathbb{Z}(n))$ — (hyper)cohomology groups.
- Finite generation, boundedness for arithmetic X: conjectures.
- Possible motivation (no pun intended):

$$E_2^{pq} = H^p(X, \mathbb{Z}(q)) \Longrightarrow K_{2q-p}(X),$$

  similar to the Atiyah–Hirzebruch spectral sequence.
- $H^i(X_{\text{ét}}, \mathbb{Z}(n))$ might be better for studying the zeta-values.

Weil-étale cohomology (Lichtenbaum, 2005)

For an arithmetic scheme X there should (!) exist abelian groups $H^i_{W,c}(X,\mathbb{Z}(0))$ and real vector spaces $H^i_W(X,\widetilde{\mathbb{R}}(0))$, $H^i_{W,c}(X,\widetilde{\mathbb{R}}(0))$ such that

1. $H^i_{W,c}(X,\mathbb{Z}(0))$ are f.g., almost all zero.
2. $H^i_{W,c}(X,\mathbb{Z}(0))\otimes\mathbb{R}\xrightarrow{\cong}H^i_{W,c}(X,\widetilde{\mathbb{R}}(0))$.
3. For a canonical class $\theta\in H^1_W(X,\widetilde{\mathbb{R}}(0))$
$$\cdots\xrightarrow{\cup\theta}H^i_{W,c}(X,\widetilde{\mathbb{R}}(0))\xrightarrow{\cup\theta}H^{i+1}_{W,c}(X,\widetilde{\mathbb{R}}(0))\xrightarrow{\cup\theta}\cdots$$
a bounded acyclic complex of f.d. vector spaces.
4. $\operatorname{ord}_{s=0}\zeta(X,s)=\sum_{i\geq 0}(-1)^i\cdot i\cdot\operatorname{rk}_{\mathbb{Z}}H^i_{W,c}(X,\mathbb{Z}(0))$.
5. $\mathbb{Z}\cdot\lambda(\zeta^*(X,0)^{-1})=\bigotimes_{i\in\mathbb{Z}}\det_{\mathbb{Z}}H^i_{W,c}(X,\mathbb{Z}(0))^{(-1)^i}$,
where $\lambda\colon\mathbb{R}\xrightarrow{\cong}\left(\bigotimes_{i\in\mathbb{Z}}\det_{\mathbb{Z}}H^i_{W,c}(X,\mathbb{Z}(0))^{(-1)^i}\right)\otimes\mathbb{R}$.

Weil-étale cohomology

- **Geisser**, 2004; **Lichtenbaum**, 2005: construction of $H^i_{W,c}(X, \mathbb{Z}(n))$ for smooth varieties over finite fields.
- **Lichtenbaum**, 2009: $H^i_{W,c}(X, \mathbb{Z}(0))$ for $X = \operatorname{Spec} \mathcal{O}_F$.
- **Morin**, 2012: $H^i_{W,c}(X, \mathbb{Z}(0))$ for proper regular arithmetic schemes.
- **Flach**, **Morin**, 2016: $H^i_{W,c}(X, \mathbb{Z}(n))$ for $n \in \mathbb{Z}$ for proper regular arithmetic schemes.
- My thesis, work in progress: $H^i_{W,c}(X, \mathbb{Z}(n))$
  - for any arithmetic scheme (makes things harder)
  - and $n = -1, -2, -3, -4, \ldots$ (makes things easier)
- Construction via the cycle complexes $\mathbb{Z}(n)$, following Flach and Morin (input: conjectures on finite generation and boundedness).

Thank you!