Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

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Par Alexey Beshenov

Valeurs zêta des schémas arithmétiques aux entiers négatifs et cohomologie Weil-étale

Sous la direction de Baptiste MORIN et Bas EDIXHOVEN

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Introduction

Let X be an **arithmetic scheme**, i.e. separated and of finite type over $\operatorname{Spec}\mathbb{Z}$. The corresponding **zeta function** is defined by the infinite product

$$\zeta(X,s) := \prod_{x\in X_0} \frac{1}{1-N(x)^{-s}},$$

where X_0 denotes the set of closed points of X, and $N(x)$ denotes the cardinality of the residue field at $x \in X_0$. This infinite product converges for $\operatorname{Re} s > \dim X$, and conjecturally, it has a meromorphic continuation to the whole complex plane. I refer to [Ser1965] for the basic results and conjectures.

This thesis is concerned with studying the special values of $\zeta(X,s)$: the goal is to interpret in cohomological terms the vanishing orders and leading Taylor coefficients at $s = n \in \mathbb{Z}$. This is a part of the program that was envisioned by Stephen Lichtenbaum and initiated in [Lic2005, Lic2009a, Lic2009b], and the conjectural underlying cohomology theory is known as Weil-étale cohomology. Later on Matthias Flach and Baptiste Morin gave a construction of Weil-étale cohomology using Bloch cycle complexes $\mathbb{Z}(n)$ to study $\zeta(X,s)$ at $s = n \in \mathbb{Z}$, see [Mor2014] and [FM2016]. Their work concerns proper regular arithmetic schemes, and the goal of this thesis is to relax these restrictions while studying the case $n < 0$.

From now on n denotes a strictly negative integer.

In chapter 0 I collect various definitions and results that are used in the constructions. Most of this material is quite standard. This chapter is lengthy, but it is needed to set up the stage.

Chapter 1 is dedicated to a construction of Weil-étale complexes

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)).$$

This will be done in two steps: first I construct complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$, which by definition give a cone of certain morphism

$$\alpha_{X,n} \colon R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

in the derived category of complexes of abelian groups:

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{aligned}$$

Then I construct yet another morphism

$$i_\infty^* \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})$$

in the derived category and declare its mapping fiber to be $R\Gamma_{W,c}(X, \mathbb{Z}(n))$:

$$\begin{aligned} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\ \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \end{aligned}$$

Finally, in chapter 2 I formulate the main conjecture. I use the regulator construction from [KLMS2006]. After reviewing the necessary preliminaries about Deligne cohomology and homology in §2.1, I define in §2.2 a morphism

$$Reg^\vee \colon R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1] \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}),$$

under the assumption that $X_\mathbb{C}$ is smooth and quasi-projective. Then $Reg^\vee$ is conjectured to be a quasi-isomorphism. This allows us to construct an ad hoc “cup product”

$$\smile \theta \colon R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}[1]$$

that gives a long exact sequence of Weil-étale cohomology groups with real coefficients

$$\begin{aligned} \cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \\ \xrightarrow{\smile\theta} H^{i+2}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots \end{aligned}$$

Then the general theory of determinants of complexes of Knudsen and Mumford implies the existence of a canonical trivialization morphism

$$\lambda \colon \mathbb{R} \xrightarrow{\cong} (\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_\mathbb{Z} \mathbb{R}.$$

Our main conjecture $\mathbf{C}(X,n)$, formulated in §2.3, says that the leading Taylor coefficient of $\zeta(X,s)$ at $s=n$ is given by

$$\lambda(\zeta^*(X,n)^{-1})\cdot\mathbb{Z}=\det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)),$$

while the corresponding vanishing order is

$$\operatorname{ord}_{s=n}\zeta(X,s)=\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n)).$$

If X is proper and regular, then this is equivalent to Conjecture 5.12 and Conjecture 5.13 from [FM2016]. In particular, it is showed in [FM2016, §5.6] that if X is projective and smooth over a number ring, then the special value conjecture is equivalent to the Tamagawa number conjecture.

Finally, I verify in §2.4 that the conjecture is compatible with the operations of taking disjoint unions of schemes, gluing schemes from an open and closed part, and passing from X to the affine space $\mathbb{A}^r_X$. This means that taking as an input the schemes for which the conjecture $\mathbf{C}(X,n)$ is known, it is possible to construct new schemes, possibly singular, for which the conjecture $\mathbf{C}(X,n)$ holds as well. This is the main unconditional outcome of the machinery developed in this thesis.

Contents

Chapter 0

Preliminaries

In this chapter we are going to fix some notation and collect several basic results which we will use later on.

Unless specified otherwise, X will denote an **arithmetic scheme**, i.e. separated, of finite type over $\operatorname{Spec}\mathbb{Z}$. Its small Zariski and étale sites will be denoted by X_{Zar} and $X_{\acute{e}t}$ respectively. By $X(\mathbb{C})$ we denote the space of complex points of X equipped with the usual analytic topology. It comes with a natural action of the Galois group $G_{\mathbb{R}} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

I start with some definitions and facts related to abelian groups in §0.1. Then in §0.2 I fix some conventions about complexes. In our constructions there will appear complexes of abelian groups of a very special kind: their cohomology is conjecturally $\mathbb{Q}/\mathbb{Z}$-dual of finitely generated abelian groups, so in §0.3 I collect some properties that are enjoyed by such complexes. We will also make use of sheaves of roots of unity, and §0.5 is dedicated to some observations about $\mu_m(\mathbb{C})$ viewed as $G_{\mathbb{R}}$-modules. We are also going to use the equivariant cohomology of sheaves on $X(\mathbb{C})$ with an action of $G_{\mathbb{R}}$. I review the basic definitions in §0.6. Then in §0.7 I recall how a sheaf on $X_{\acute{e}t}$ gives rise to a $G_{\mathbb{R}}$-equivariant sheaf on $X(\mathbb{C})$. In §0.8 I recall the definitions of cohomology with compact support for sheaves on $X_{\acute{e}t}$ and $X(\mathbb{C})$, and in §0.9 I review a slight modification of cohomology with compact support on $X_{\acute{e}t}$ needed for arithmetic duality theorems, which will show up in §1.3. Then in §0.10 I sketch a proof that for any arithmetic scheme X, the cohomology groups $H^i_c(X(\mathbb{C}), \mathbb{Z})$ are finitely generated (this seems to be very standard, but I could not find a reference). Finally, §0.11 is dedicated to an overview of Bloch's cycle complexes.

0.1 Abelian groups

Let A be an abelian group. Then A_{tor} denotes the maximal torsion subgroup of A and A_{cotor} denotes the group A/A_{tor}. Similarly, A_{div} denotes the maximal divisible subgroup of A and A_{codiv} denotes the group A/A_{div}, and we have short exact sequences

$$0 \to A_{tor} \to A \to A_{cotor} \to 0,$$
$$0 \to A_{div} \to A \to A_{codiv} \to 0.$$

Note that the image of a divisible group is divisible, so that a group homomorphism $f\colon A \to B$ induces functorially a homomorphism of divisible groups $f_{div}\colon A_{div} \to B_{div}$. If A is a divisible group, then

$$\mathrm{Hom}_{\mathbf{Ab}}(A, B) \cong \mathrm{Hom}_{\mathbf{DivAb}}(A, B_{div}),$$

so that taking the maximal divisible subgroup $(-)_{div}\colon \mathbf{Ab} \to \mathbf{DivAb}$ is right adjoint to the inclusion $\mathbf{DivAb} \hookrightarrow \mathbf{Ab}$.

For the group of homomorphisms $A \to B$ between two abelian groups, we will write simply $\mathrm{Hom}(A, B)$. For $m = 1, 2, 3, \ldots$ we denote by

$${}_mA := \ker(A \xrightarrow{m} A) \cong \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, A)$$

the m-torsion subgroup of A, and dually,

$$A_m := \mathrm{coker}(A \xrightarrow{m} A) = A/mA.$$

We have thus an exact sequence

$$0 \to {}_mA \to A \xrightarrow{\times m} A \to A_m \to 0$$

The abelian group $\mathbb{Q}/\mathbb{Z}$ is divisible, hence injective, meaning that the contravariant functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is exact. For the infinite cyclic group we have trivially

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z},$$

and for finite cyclic groups

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) &\cong {}_m(\mathbb{Q}/\mathbb{Z}) \\ &= \{[0/m],\ [1/m],\ [2/m],\ \ldots,\ [m-1/m]\} \cong \mathbb{Z}/m\mathbb{Z}.\end{aligned}$$

It follows that if A is a finitely generated abelian group, then $A \cong \mathbb{Z}^{\oplus r} \oplus A_{tor}$, where A_{tor} is the finite maximal torsion subgroup in A, and

$$\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus A_{tor}.$$

Of course, this isomorphism is not canonical, as it requires a choice of generators.

0.1.1. Definition. If $B \cong \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ for a finitely generated abelian group A, we say that B is **of cofinite type**.

0.1.2. Observation. *If A is a finitely generated abelian group, then there is a canonical isomorphism*

$$\varinjlim_m \mathrm{Hom}(A/mA, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}).$$

Proof. This isomorphism is induced by $A \to A/mA$, and then applying the functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and $\varinjlim_m$. It comes from the following easy observation: as $\mathbb{Q}/\mathbb{Z}$ is a torsion group, if A is finitely generated, any homomorphism $A \to \mathbb{Q}/\mathbb{Z}$ is killed by some m, hence factors through $A/mA \to \mathbb{Q}/\mathbb{Z}$. ∎

0.1.3. Lemma. *Denote $(-)^D := \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$. Let A and B be finitely generated abelian groups and let A^D and B^D be the corresponding groups of cofinite type. Then every extension of B^D by A^D is again a group of cofinite type. Namely, any such extension is equivalent to*

$$0 \to A^D \to C^D \to B^D \to 0 \tag{0.1.1}$$

where

$$0 \to B \to C \to A \to 0 \tag{0.1.2}$$

is an extension of A by B.

The statement seems trivial, especially because $\mathrm{Ext}(A, B)$ and $\mathrm{Ext}(B^D, A^D)$ are easily seen to be isomorphic finite groups. However, there is one subtle issue: it is not obvious why nonequivalent extensions (0.1.2) cannot for some reason give equivalent extensions (0.1.1). Indeed, between groups of cofinite type, there are many homomorphisms that are not induced from the corresponding finitely generated groups; for example,

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \quad \text{while} \quad \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}^D, \mathbb{Z}^D) \cong \widehat{\mathbb{Z}}. \tag{0.1.3}$$

A priori, these extra homomorphisms could give weird equivalences of extensions. This is not the case, but we need to be a little bit more careful to justify that.

Proof. Consider the category $\mathbf{Ab}_{ft}$ of finitely generated abelian groups. It is a full abelian subcategory of the category $\mathbf{Ab}$. The contravariant functor

$$(-)^D := \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})\colon \mathbf{Ab}_{ft}^{\circ} \to \mathbf{Ab}.$$

is exact and faithful, but it is very far from being full, as we observed in (0.1.3). Let us denote the image of the functor $(-)^D$ by $\mathbf{Ab}_{cft}$. It is the category whose objects are groups of cofinite type A^D for some finitely generated A, and morphisms $B^D \to A^D$ in $\mathbf{Ab}_{cft}$ are induced by morphisms $A \to B$ of finitely generated groups. This means that $(-)^D$ restricts to an (anti)equivalence of abelian categories

$$(0.1.4) \qquad (-)^D := \operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})\colon \mathbf{Ab}_{ft}^{\circ} \xrightarrow{\cong} \mathbf{Ab}_{cft}.$$

The category $\mathbf{Ab}_{ft}$ has enough projective objects (and no nontrivial injective objects). Dually, $\mathbf{Ab}_{cft}$ has enough injective objects: they are $\mathbb{Q}/\mathbb{Z}$-dual to the projective objects in $\mathbf{Ab}_{ft}$:

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "\widetilde{f}"'] \\ A \arrow[r, two heads] \arrow[ur, phantom, "\exists"] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & P^D \\ A^D \arrow[ur, dashed, "\widetilde{f^D}"] & B^D \arrow[l, tail] \arrow[u, "f^D"'] \arrow[ul, phantom, "\exists"] \end{tikzcd}$$

Now assume that for some finitely generated groups A and B we want to calculate

$$\operatorname{Ext}^1_{\mathbf{Ab}}(A, B) \cong R^1 \operatorname{Hom}_{\mathbf{Ab}}(-, B)(A) = R^1 \operatorname{Hom}_{\mathbf{Ab}_{ft}}(-, B)(A) \cong \operatorname{Ext}^1_{\mathbf{Ab}_{ft}}(A, B).$$

To do this, we may pick a projective resolution $P_\bullet \twoheadrightarrow A$, and then calculate the cohomology group $H^1 \operatorname{Hom}(P_\bullet, B)$. Note that we may build this projective resolution from finitely generated groups, i.e. inside the category $\mathbf{Ab}_{ft}$. Then thanks to the (anti)equivalence of categories (0.1.4), we have

$$(0.1.5) \qquad \operatorname{Ext}^1_{\mathbf{Ab}}(A, B) \cong \operatorname{Ext}^1_{\mathbf{Ab}_{ft}}(A, B) \cong \operatorname{Ext}^1_{\mathbf{Ab}_{cft}}(B^D, A^D).$$

The group

$$\operatorname{Ext}^1_{\mathbf{Ab}_{cft}}(B^D, A^D) \cong R^1 \operatorname{Hom}_{\mathbf{Ab}_{cft}}(B^D, -)(A^D)$$

may be calculated by taking the same resolution $P_\bullet \twoheadrightarrow A$, dualizing it to obtain an injective resolution $A^D \rightarrowtail P_\bullet^D$ by groups of cofinite type, and then calculating $H^1 \operatorname{Hom}_{\mathbf{Ab}_{cft}}(B^D, P_\bullet^D)$. Note that $\operatorname{Hom}_{\mathbf{Ab}_{cft}}(B^D, P_\bullet^D)$ is a subcomplex in $\operatorname{Hom}_{\mathbf{Ab}}(B^D, P_\bullet^D)$, and we have the corresponding homomorphism on H^1

$$(0.1.6) \qquad \operatorname{Ext}^1_{\mathbf{Ab}_{cft}}(B^D, A^D) \to \operatorname{Ext}^1_{\mathbf{Ab}}(B^D, A^D).$$

I claim that it is an isomorphism. Indeed, by additivity of $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$, it is enough to see this for the only interesting case $A = \mathbb{Z}/m\mathbb{Z}$ and $B = \mathbb{Z}$. The projective resolution

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{1 \mapsto [1]} \mathbb{Z}/m\mathbb{Z} \to 0$$

gives us the corresponding injective resolution of $\mathbb{Z}/m\mathbb{Z}^D \cong \mathbb{Z}/m\mathbb{Z}$:

$$0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{[1]\mapsto[1/m]} \mathbb{Q}/\mathbb{Z} \xrightarrow{\times m} \mathbb{Q}/\mathbb{Z} \to 0$$

After applying $\operatorname{Hom}_{\mathcal{A}}(\mathbb{Z}^D, -)$ for $\mathcal{A} = \mathbf{Ab}_{cft}, \mathbf{Ab}$, we get two complexes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times m"] \arrow[d, tail] & \mathbb{Z} \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \widehat{\mathbb{Z}} \arrow[r, "\times m"] & \widehat{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

On H^1 this indeed induces an isomorphism $\mathbb{Z}/m\mathbb{Z} \to \widehat{\mathbb{Z}}/m\widehat{\mathbb{Z}} \cong \mathbb{Z}/m\mathbb{Z}$. Combining the isomorphism (0.1.6) with (0.1.5), we obtain an isomorphism

$$\operatorname{Ext}^1_{\mathbf{Ab}}(A, B) \cong \operatorname{Ext}^1_{\mathbf{Ab}}(B^D, A^D).$$

It remains to pass to the Yoneda Ext, which I suggest to denote for the moment by $\operatorname{YExt}^1_{\mathcal{A}}(A, B)$, and which corresponds to the equivalence classes of extensions

$$0 \to B \to C \to A \to 0$$

with respect to the Baer sum. If we have enough projectives or injectives in $\mathcal{A}$, so that $\operatorname{Ext}^1_{\mathcal{A}}(A, B)$ exists, then we have an isomorphism of abelian groups

$$\operatorname{YExt}^1_{\mathcal{A}}(A, B) \cong \operatorname{Ext}^1_{\mathcal{A}}(A, B)$$

—see e.g. [Wei1994, §3.4]. In our situation, this gives an isomorphism

$$\begin{gathered}
\operatorname{YExt}^1_{\mathbf{Ab}}(A, B) \xrightarrow{\cong} \operatorname{YExt}^1_{\mathbf{Ab}}(B^D, A^D), \\
[B \rightarrowtail C \twoheadrightarrow A] \mapsto [A^D \rightarrowtail C^D \twoheadrightarrow B^D]
\end{gathered}$$

■

0.1.4. Example. If T is a finite abelian group, then

$$\operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, T) \cong \operatorname{Ext}(T, \mathbb{Z}) \cong T.$$

Indeed, by additivity of $\operatorname{Ext}(-,-)$, it is enough to check this for cyclic groups $T \cong \mathbb{Z}/m\mathbb{Z}$, and in this case, after applying $\operatorname{Hom}(-, \mathbb{Z})$ to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

we obtain

$$0 \to \underbrace{\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})}_{=0} \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z}$$
$$\to \mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \to \underbrace{\mathrm{Ext}(\mathbb{Z}, \mathbb{Z})}_{=0} \to \underbrace{\mathrm{Ext}(\mathbb{Z}, \mathbb{Z})}_{=0} \to 0$$

In particular, for prime p, the corresponding p nonequivalent extensions of $\mathbb{Q}/\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ arise as follows. First, there is the split extension

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

which is dual to the extension

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

Then the remaining $p-1$ extensions are of the form

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{[1]\mapsto[m/p]} \mathbb{Q}/\mathbb{Z} \xrightarrow{\times p} \mathbb{Q}/\mathbb{Z} \to 0$$

where $m = 1, 2, \ldots, p-1$. Here we identify $\mathbb{Z}/p\mathbb{Z}$ with the cyclic subgroup $\left\{0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}\right\} \subset \mathbb{Q}/\mathbb{Z}$. These extensions are dual to

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{1\mapsto[m]} \mathbb{Z}/p\mathbb{Z} \to 0$$

They are not equivalent for different m, because if we have a commutative diagram

$$\begin{tikzcd}
 & & \mathbb{Q}/\mathbb{Z} \arrow[dd, "\cong"] \arrow[dr, "\times p"] & & \\
0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[ur, "{[1]\mapsto[m_1/p]}"] \arrow[dr, "{[1]\mapsto[m_2/p]}"'] & & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \\
 & & \mathbb{Q}/\mathbb{Z} \arrow[ur, "\times p"'] & &
\end{tikzcd}$$

then $m_1 = m_2$. ▲

0.2 Complexes

Let us recall a couple of constructions from homological algebra. For an abelian category $\mathcal{A}$ we denote by $\mathbf{C}(\mathcal{A})$ the category of cohomological complexes in $\mathcal{A}$, by $\mathbf{K}(\mathcal{A})$ the corresponding homotopy category, and by $\mathbf{D}(\mathcal{A})$ the derived category.

For a complex $C^\bullet$ and $p \in \mathbb{Z}$, the shifted complex $C^\bullet[p]$ is defined by

$$(C^\bullet[p])^i := C^{i+p}, \quad d^i_{C^\bullet[p]} := (-1)^p\, d^{i+p}.$$

With this convention, $H^i(C^\bullet[p]) = H^{i+p}(C^\bullet)$. (Note that some sources, e.g. [Wei1994, 1.2.4], use another renumbering $(C^\bullet[p])^i := C^{i-p}$.)

0.2.1. Definition. A (cohomological) **double complex** $(C^{\bullet\bullet}, d_h^{\bullet\bullet}, d_v^{\bullet\bullet})$ is given by objects $C^{i,j} \in \mathcal{A}$ for $i,j \in \mathbb{Z}$, **horizontal differentials**

$$d_h^{i,j}\colon C^{i,j} \to C^{i+1,j},$$

and **vertical differentials**

$$d_v^{i,j}\colon C^{i,j} \to C^{i,j+1},$$

such that for all $i,j \in \mathbb{Z}$

$$d_v^{i+1,j} \circ d_h^{i,j} + d_h^{i,j+1} \circ d_v^{i,j} = 0; \tag{0.2.1}$$

that is, we have a diagram with anti-commutative squares

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{i-1,j+1} \arrow[u] \arrow[r] & C^{i,j+1} \arrow[u] \arrow[r, "d_h^{i,j+1}"] & C^{i+1,j+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{i-1,j} \arrow[u] \arrow[r] & C^{i,j} \arrow[u, "d_v^{i,j}"] \arrow[r, "d_h^{i,j}"] & C^{i+1,j} \arrow[u, "d_v^{i+1,j}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{i-1,j-1} \arrow[u] \arrow[r] & C^{i,j-1} \arrow[u] \arrow[r] & C^{i+1,j-1} \arrow[u] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Assume that in $\mathcal{A}$ exist arbitrary products $\prod_i A_i$ and coproducts $\bigoplus_i A_i$. Then the corresponding **total complexes** (with respect to direct sum and product) are given by

$$(\operatorname{Tot}^\oplus C^{\bullet\bullet})^m := \bigoplus_{i+j=m} C^{i,j}, \quad (\operatorname{Tot}^\Pi C^{\bullet\bullet})^m := \prod_{i+j=m} C^{i,j},$$

together with the obvious differentials $d^m\colon (\operatorname{Tot} C^{\bullet\bullet})^m \to (\operatorname{Tot} C^{\bullet\bullet})^{m+1}$ defined via $d_h^{\bullet\bullet}$ and $d_v^{\bullet\bullet}$. The identity $d^{m+1} \circ d^m = 0$ is satisfied thanks to the condition (0.2.1).

Note that if $C^{i,j} = 0$ for $i \ll 0$ and for $j \ll 0$, then for each m there are only finitely many nonzero objects $C^{i,j}$ such that $i + j = m$, and in this case $\operatorname{Tot}^{\oplus} C^{\bullet\bullet} = \operatorname{Tot}^{\Pi} C^{\bullet\bullet}$.

0.2.2. Definition. Let $(A_\bullet, d^A_\bullet)$ be a homological complex and $(B^\bullet, d^\bullet_B)$ a cohomological complex. Then the corresponding **Hom double complex** $\operatorname{Hom}^{\bullet\bullet}(A_\bullet, B^\bullet)$ is the double complex of abelian groups given by

$$\operatorname{Hom}^{i,j}(A_\bullet, B^\bullet) := \operatorname{Hom}_{\mathcal{A}}(A_i, B^j),$$

together with the differentials for $f \in \operatorname{Hom}_{\mathcal{A}}(A_i, B^j)$

$$\begin{aligned} d_h^{i,j} f &:= f \circ d^A_{i+1}, \\ d_v^{i,j} f &:= (-1)^{i+j+1}\, d_B^j \circ f. \end{aligned} \tag{0.2.2}$$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & \operatorname{Hom}(A_{i-1}, B^{j+1}) & \to & \operatorname{Hom}(A_i, B^{j+1}) & \to & \operatorname{Hom}(A_{i+1}, B^{j+1}) & \to & \cdots \\
 & & \uparrow & & {\scriptstyle d_v^{i,j}}\uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \operatorname{Hom}(A_{i-1}, B^{j}) & \longrightarrow & \operatorname{Hom}(A_i, B^{j}) & \xrightarrow{d_h^{i,j}} & \operatorname{Hom}(A_{i+1}, B^{j}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & \operatorname{Hom}(A_{i-1}, B^{j-1}) & \to & \operatorname{Hom}(A_i, B^{j-1}) & \to & \operatorname{Hom}(A_{i+1}, B^{j-1}) & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The sign in (0.2.2) is introduced to make the squares anti-commute, turning $\operatorname{Hom}^{\bullet\bullet}(A_\bullet, B^\bullet)$ into a double complex in the sense of 0.2.1.

0.2.3. Definition. Let $(A^\bullet, d^\bullet_A)$ and $(B^\bullet, d^\bullet_B)$ be two cohomological complexes. Then we may turn $A^\bullet$ into a homological complex $A_\bullet$ by setting $A_i := A^{-i}$ and $d^A_i := d_A^{-i}\colon A_i \to A_{i-1}$. The complex

$$\operatorname{Hom}^\bullet(A^\bullet, B^\bullet) := \operatorname{Tot}^{\Pi} \operatorname{Hom}^{\bullet\bullet}(A_\bullet, B^\bullet)$$

is called the **Hom complex**.

0.3 Derived category of abelian groups

Most of the time we are going to work in the derived category $\mathbf{D}(\mathbf{Ab})$ of complexes of abelian groups, and occasionally the derived category of $\mathbf{D}(\mathbb{R}\textbf{-Vect})$

of complexes of real vector spaces. The canonical reference for derived categories is Verdier's thesis [Verdier-thèse], and in particular I am going to use Verdier's original axioms (TR1)–(TR4).

It is rather easy to describe how objects and morphisms in the category $\mathbf{D}(\mathbf{Ab})$ look like, thanks to the fact that $\operatorname{Ext}^i_{\mathbb{Z}}(-,-) = 0$ for $i > 1$. Let us recall the general (well-known) result.

0.3.1. Lemma. *Let $\mathcal{A}$ be a* ***hereditary abelian category****, i.e. an abelian category such that*

$$\operatorname{Ext}^i_{\mathcal{A}}(A,B) = 0 \quad \text{for all } A,B \in \mathcal{A},\ i > 1$$

*(when $\mathcal{A} = R$-**Mod**, this condition is equivalent to R being a* ***hereditary ring****; in particular, $\mathbb{Z}$ and any principal ideal domain is hereditary).*

1) *In the derived category $\mathbf{D}(\mathcal{A})$ every complex $A^\bullet$ is isomorphic to the complex*

$$\cdots \to H^{i-1}(A^\bullet) \xrightarrow{0} H^i(A^\bullet) \xrightarrow{0} H^{i+1}(A^\bullet) \to \cdots$$

that is,

$$A^\bullet \cong \bigoplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i] \cong \prod_{i\in\mathbb{Z}} H^i(A^\bullet)[-i].$$

2) *The morphisms in $\mathbf{D}(\mathcal{A})$ are given by*

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet,B^\bullet) \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(H^i(A^\bullet),H^i(B^\bullet)) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Ext}^1_{\mathcal{A}}(H^i(A^\bullet),H^{i-1}(B^\bullet)).$$

Proof. For the first part, for each $i \in \mathbb{Z}$ let us consider the short exact sequence

$$0 \to \ker d^{i-1} \to A^{i-1} \xrightarrow{p} \operatorname{im} d^{i-1} \to 0$$

Applying the functor $\operatorname{Hom}_{\mathcal{A}}(H^i(A^\bullet),-)$ gives us a long exact sequence of Yoneda Exts

$$\begin{aligned}\cdots \to \operatorname{Ext}^1_{\mathcal{A}}(H^i(A^\bullet),\ker d^{i-1}) &\to \operatorname{Ext}^1_{\mathcal{A}}(H^i(A^\bullet),A^{i-1}) \\ &\xrightarrow{p_*} \operatorname{Ext}^1_{\mathcal{A}}(H^i(A^\bullet),\operatorname{im} d^{i-1}) \to \operatorname{Ext}^2_{\mathcal{A}}(H^i(A^\bullet),\ker d^{i-1}) \to \cdots\end{aligned}$$

where the last Ext vanishes by our assumption on $\mathcal{A}$, and therefore p_* is surjective, which in particular means that the class of the short exact sequence

$$0 \to \operatorname{im} d^{i-1} \to \ker d^i \to H^i(A^\bullet) \to 0$$

lies in the image in p_*, so that there exists an object B^i sitting in the following morphism of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner"] & B^i \arrow[r] \arrow[d] & H^i(A^\bullet) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \operatorname{im} d^{i-1} \arrow[r] & \ker d^i \arrow[r] & H^i(A^\bullet) \arrow[r] & 0 \end{tikzcd}$$

This gives us morphisms of complexes

$$\begin{tikzcd} & [\underset{i-1}{A^{i-1}} \to \underset{i}{B^i}] \arrow[dl] \arrow[dr] & \\ A^\bullet & & H^i(A^\bullet)[-i] \end{tikzcd}$$

that induce isomorphisms in cohomology in degree i:

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-2} \arrow[r] & A^{i-1} \arrow[r] & A^i \arrow[r] & A^{i+1} \arrow[r] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & A^{i-1} \arrow[r] \arrow[u, "\mathrm{id}"] \arrow[d] & B^i \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^i(A^\bullet) \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Passing to direct sums of the complexes $H^i(A^\bullet)[-i]$ and $[\underset{i-1}{A^{i-1}} \to \underset{i}{B^i}]$ gives us quasi-isomorphisms that form the desired isomorphism in $\mathbf{D}(\mathcal{A})$:

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ A^\bullet & & \bigoplus_{i \in \mathbb{Z}} H^i(A^\bullet)[-i] \end{tikzcd}$$

We note that $\bigoplus_{i \in \mathbb{Z}} H^i(A^\bullet)[-i]$ has the universal property of both product and coproduct in the category of complexes.

Now for the second part, we note that since by our assumptions on $\mathcal{A}$,

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A, B[i]) = \begin{cases} \operatorname{Hom}_{\mathcal{A}}(A, B), & i = 0, \\ \operatorname{Ext}^1_{\mathcal{A}}(A, B), & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

we have by the calculation in 1),

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet) &\cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(\bigoplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i], \prod_{j\in\mathbb{Z}} H^j(B^\bullet)[-j]) \\
&\cong \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(H^i(A^\bullet), H^j(B^\bullet)[i-j]) \\
&\cong \prod_{i\in\mathbb{Z}} \Big(\operatorname{Hom}_{\mathcal{A}}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \operatorname{Ext}^1_{\mathcal{A}}(H^i(A^\bullet), H^{i-1}(B^\bullet))\Big).
\end{aligned}$$

■

0.3.2. Remark. One can also obtain information about $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ using the following **hyperext spectral sequence**:

$$E_2^{pq} = \prod_{i\in\mathbb{Z}} \operatorname{Ext}^p_{\mathcal{A}}(H^i(A^\bullet), H^{q+i}(B^\bullet)) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$$

(see e.g. [Verdier-thèse, Chapitre III, §4.6.10] and [Wei1994, §5.7.9]). For a hereditary category $\operatorname{Ext}^p_{\mathcal{A}} = 0$, unless $p = 0, 1$, and this spectral sequence consists of two columns and therefore gives us short exact sequences

$$\begin{aligned}
0 \to \prod_{i\in\mathbb{Z}} \operatorname{Ext}^1_{\mathcal{A}}(H^i(A^\bullet), H^{i-1}(B^\bullet)) &\to \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet) \\
&\to \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(H^i(A^\bullet), H^i(B^\bullet)) \to 0
\end{aligned}$$

However, one should be careful with boundedness of $A^\bullet$ and $B^\bullet$ to make sure that the spectral sequence exists.

Recall that a complex of abelian groups $C^\bullet$ is called **perfect** if it is quasi-isomorphic to a bounded complex of finitely generated free (= projective) abelian groups. This is the same as asking $H^i(C^\bullet)$ to be finitely generated abelian groups, and $H^i(C^\bullet) = 0$ for all but finitely many i. In §1.5 we are going to construct certain complexes $R\Gamma_{fg}(X, \mathbb{Z}(n))$ that are *almost* perfect, in the sense that their cohomology groups $H^i_{fg}(X, \mathbb{Z}(n))$ are finitely generated, vanish for $i \ll 0$, and for $i \gg 0$ they are finite 2-torsion (that is, killed by multiplication by 2). Let us introduce the following notion.

0.3.3. Definition. Let $C^\bullet$ be an object in $\mathbf{D}(\mathbf{Ab})$. We say that $C^\bullet$ is **almost perfect** if

1) $H^i(C^\bullet)$ are finitely generated groups,

2) $H^i(C^\bullet) = 0$ for $i \ll 0$,

3) $H^i(C^\bullet)$ is 2-torsion for $i \gg 0$.

I warn the reader that this terminology was invented by myself and serves only to simplify the exposition.

0.3.4. Lemma.

1) *If $C^\bullet$ and $C'^\bullet$ are almost perfect, then the group* $\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(C^\bullet, C'^\bullet)$ *has no nontrivial divisible subgroups.*

2) *If $A^\bullet$ is a complex such that $H^i(A^\bullet)$ are finite dimensional $\mathbb{Q}$-vector spaces and $C^\bullet$ is a complex such that $H^i(C^\bullet)$ are finitely generated abelian groups, then the group* $\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, C^\bullet)$ *is divisible.*

Proof. By 0.3.1 we have

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(C^\bullet, C'^\bullet) \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(C^\bullet), H^i(C'^\bullet)) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(C^\bullet), H^{i-1}(C'^\bullet)).$$

Note that by our assumptions, both groups $\prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(C^\bullet), H^i(C'^\bullet))$ and $\prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(C^\bullet), H^{i-1}(C'^\bullet))$ will be of the form $G \oplus T$, where G is a finitely generated abelian group and T is 2-torsion. Assume now that some element $x \in \operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(C^\bullet, C'^\bullet)$ is divisible by all powers of 2. If it lies in the finitely generated part, then $x = 0$; if it lies in the 2-torsion part, then again $x = 0$.

Similarly, in part 2), we have

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, C^\bullet) \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(C^\bullet)) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(A^\bullet), H^{i-1}(C^\bullet)).$$

Now by our assumptions $\operatorname{Hom}(H^i(A^\bullet), H^i(C^\bullet)) = 0$ for all i. Then each group $\operatorname{Ext}(H^i(A^\bullet), H^{i-1}(C^\bullet))$ is a direct sum of finitely many groups isomorphic to $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ and $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$, and $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ is divisible while $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z}) = 0$. This means that the group $\operatorname{Ext}(H^i(A^\bullet), H^{i-1}(C^\bullet))$ is divisible for each i, and hence their direct product over i is divisible. ■

Recall that the axiom (TR1) tells us that every morphism $v\colon A^\bullet \to B^\bullet$ may be completed to a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$. Here $C^\bullet$ is called the **cone** of u. The axiom (TR3) tells that for every commutative diagram with distinguished rows

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A^\bullet[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\
A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'^\bullet[1]
\end{array} \tag{0.3.1}$$

there exists some morphism $h\colon C^\bullet \to C'^\bullet$ giving a morphism of distinguished triangles

$$\begin{tikzcd} A^\bullet \arrow[r, "u"] \arrow[d, "f"] & B^\bullet \arrow[r, "v"] \arrow[d, "g"] & C^\bullet \arrow[r, "w"] \arrow[d, dashed, "\exists! h"] & A^\bullet[1] \arrow[d, "f[1]"] \\ A'^\bullet \arrow[r, "u'"] & B'^\bullet \arrow[r, "v'"] & C'^\bullet \arrow[r, "w'"] & A'^\bullet[1] \end{tikzcd} \tag{0.3.2}$$

The cone $C^\bullet$ in (TR1) and the morphism h in (TR3) are neither unique nor canonical. Two different cones of the same morphism are necessarily isomorphic, but the isomorphism between them is not unique, because it is provided by (TR3). This is a well-known issue with the derived category formalism, and in the present text we are going to encounter some problems related to it. For now, let us recall a useful standard argument which shows that at least in some special cases, things are uniquely defined.

0.3.5. Observation ((TR3) and (TR1) with uniqueness; $\approx$ [BBD1982, Proposition 1.1.9, Corollaire 1.1.10]). *Consider the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$.*

1) *For a commutative diagram (0.3.1), assume that the homomorphism of abelian groups*
$$w^*\colon \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \to \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet)$$
induced by w is trivial. Then there exists a unique morphism $h\colon C^\bullet \to C'^\bullet$ giving a morphism of triangles (0.3.2).

2) *For a distinguished triangle $A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$, assume that for any other cone $C'^\bullet$ of u the morphism w^* is trivial. Then in fact the cone of u is unique up to a unique isomorphism.*

Proof. In 1), the existence of $C^\bullet \to C'^\bullet$ is the axiom (TR3), and the interesting part is uniqueness. Since $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(-, C'^\bullet)$ is a cohomological functor, applied to the first distinguished triangle, it gives us an exact sequence of abelian groups

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet[1], C'^\bullet) \xrightarrow{w^*} \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(C^\bullet, C'^\bullet) \xrightarrow{v^*} \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(B^\bullet, C'^\bullet).$$

If $w^* = 0$, we conclude that v^* is a monomorphism. This means that there is a unique morphism h such that $h \circ v = v' \circ g$. Now in 2), if $C^\bullet$ and $C'^\bullet$ are two different cones of u, we have a commutative diagram

$$\begin{tikzcd} A^\bullet \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & B^\bullet \arrow[r, "v"] \arrow[d, "\mathrm{id}"] & C^\bullet \arrow[r, "w"] \arrow[d, dashed] & A^\bullet[1] \arrow[d, "\mathrm{id}"] \\ A^\bullet \arrow[r, "u'"] & B^\bullet \arrow[r, "v'"] & C'^\bullet \arrow[r, "w'"] & A^\bullet[1] \end{tikzcd}$$

As always, by the "triangulated 5-lemma", the dashed arrow is an isomorphism, and it is unique thanks to 1). ■

Here is a particular case that we are going to use.

0.3.6. Corollary. *Consider the derived category* $\mathbf{D}(\mathbf{Ab})$.

1) *Suppose we have a commutative diagram with distinguished rows*

$$\begin{tikzcd} A^\bullet \arrow[r, "u"] \arrow[d, "f"] & B^\bullet \arrow[r, "v"] \arrow[d, "g"] & C^\bullet \arrow[r, "w"] & A^\bullet[1] \\ A'^\bullet \arrow[r, "u'"] & B'^\bullet \arrow[r, "v'"] & C'^\bullet \arrow[r, "w'"] & A'^\bullet[1] \end{tikzcd}$$

where $A^\bullet$ *is a complex such that* $H^i(A^\bullet)$ *are finite dimensional* $\mathbb{Q}$*-vector spaces and* $C^\bullet$ *and* $C'^\bullet$ *are almost perfect complexes. Then there exists a unique (!) morphism* $h\colon C^\bullet \to C'^\bullet$ *giving a morphism of triangles*

$$\begin{tikzcd} A^\bullet \arrow[r, "u"] \arrow[d, "f"] & B^\bullet \arrow[r, "v"] \arrow[d, "g"] & C^\bullet \arrow[r, "w"] \arrow[d, dashed, "\exists!h"] & A^\bullet[1] \arrow[d, "f[1]"] \\ A'^\bullet \arrow[r, "u'"] & B'^\bullet \arrow[r, "v'"] & C'^\bullet \arrow[r, "w'"] & A'^\bullet[1] \end{tikzcd}$$

2) *For a distinguished triangle*

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A^\bullet[1]$$

assume that $A^\bullet$ *is a complex such that* $H^i(A^\bullet)$ *are finite dimensional* $\mathbb{Q}$*-vector spaces and* $C^\bullet$ *is an almost perfect complex. Then the cone of* u *is unique up to a unique isomorphism.*

Proof. In this situation, according to 0.3.4, the group $\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(C^\bullet, C'^\bullet)$ has no nontrivial divisible subgroups and $\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet[1], C'^\bullet)$ is divisible. This means that there are no nontrivial homomorphisms

$$\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet[1], C'^\bullet) \to \mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(C^\bullet, C'^\bullet)$$

and we may apply 0.3.5. ■

We are going to encounter certain complexes whose cohomology groups are of cofinite type, i.e. $\mathbb{Q}/\mathbb{Z}$-dual of finitely generated abelian groups. Again, they will be bounded below, but may have 2-torsion in higher degrees. For this we introduce a definition similar to 0.3.3.

0.3.7. Definition. Let $A^\bullet$ be an object in $\mathbf{D}(\mathbf{Ab})$. We say that $A^\bullet$ is **almost of cofinite type** if

1) $H^i(A^\bullet)$ are groups of cofinite type for all i,

2) $H^i(A^\bullet) = 0$ for $i \ll 0$,

3) $H^i(A^\bullet)$ is 2-torsion for $i \gg 0$ (in fact, finite 2-torsion according to 1)).

0.3.8. Observation. *Suppose that $A^\bullet$ and $B^\bullet$ are almost of cofinite type. Then a morphism $f\colon A^\bullet \to B^\bullet$ is torsion in $\mathbf{D}(\mathbf{Ab})$ (i.e. a torsion element in the group $\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, B^\bullet)$, i.e. $f \otimes \mathbb{Q} = 0$) if and only if the morphisms*

$$H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$$

are torsion, that is, they are trivial on the maximal divisible subgroups:

$$(H^i(f)_{div}\colon H^i(A^\bullet)_{div} \to H^i(B^\bullet)_{div}) = 0.$$

Proof. By 0.3.1 we have

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, B^\bullet) \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet)).$$

As the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ are of the form $(\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus T$, where T is finite, we have

$$\begin{aligned}\operatorname{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r} \oplus T, (\mathbb{Q}/\mathbb{Z})^{\oplus r'} \oplus T') \cong&\\ \underbrace{\operatorname{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})}_{=0} \;\oplus\; &\operatorname{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, T')\\ \oplus\; \underbrace{\operatorname{Ext}(T, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})}_{=0} \;\oplus\; &\operatorname{Ext}(T, T'),\end{aligned}$$

where $\operatorname{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})$ and $\operatorname{Ext}(T, (\mathbb{Q}/\mathbb{Z})^{\oplus r'})$ are trivial because $\mathbb{Q}/\mathbb{Z}$ is a divisible group; then $\operatorname{Ext}((\mathbb{Q}/\mathbb{Z})^{\oplus r}, T') \cong \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, T')^{\oplus r} \cong T'^{\oplus r}$ by 0.1.4, and $\operatorname{Ext}(T, T')$ is also finite, being a direct sum of

$$\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z}.$$

For $i \gg 0$, the groups $H^i(A^\bullet)$ and $H^{i-1}(B^\bullet)$ will be finite 2-torsion, and therefore $\operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ will be finite 2-torsion as well. It follows that the whole product $\prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$ is of the form $G \oplus T$, where G is finite and T is possibly infinite 2-torsion. We have

$$(G \oplus T) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Similarly, the group $\prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet))$ will consist of some part of the form $\widehat{\mathbb{Z}}^{\oplus r} \oplus G$, where G is finite, and some 2-torsion part, which is killed by tensoring with $\mathbb{Q}$. It follows that there is an isomorphism

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, B^\bullet) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

After unwinding the proof of 0.3.1, one sees that this arrow is what it should be:

$$f \otimes \mathbb{Q} \mapsto (H^i(f) \otimes \mathbb{Q})_{i\in\mathbb{Z}}.$$

■

0.3.9. Observation. *If $A^\bullet$ is a complex of $\mathbb{Q}$-vector spaces and $B^\bullet$ is a complex almost of cofinite type, then there is an isomorphism of abelian groups*

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, B^\bullet) \xrightarrow{\cong} \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)),$$
$$f \mapsto (H^i(f))_{i\in\mathbb{Z}}.$$

Proof. I claim that in the formula 0.3.1

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(A^\bullet, B^\bullet) \cong \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H^i(A^\bullet), H^i(B^\bullet)) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Ext}(H^i(A^\bullet), H^{i-1}(B^\bullet))$$

the summand with Ext groups vanishes. Indeed, each group $H^{i-1}(B^\bullet)$ is of the form $\mathbb{Q}/\mathbb{Z}^{\oplus r} \oplus T$, where $\mathbb{Q}/\mathbb{Z}$ is injective, hence $\operatorname{Ext}(-, \mathbb{Q}/\mathbb{Z}) = 0$, and T is a finite torsion group, hence $\operatorname{Ext}(V, T) = 0$ if V is a $\mathbb{Q}$-vector space. ■

0.4 Determinants of complexes

We are going use determinants of complexes defined by Knudsen and Mumford. The reader may consult [GKZ1994, Appendix A] for a nice introduction and the original paper [KM1976] for the technical details.

For a perfect complex of R-modules $P^\bullet$, or in general for a perfect complex in the derived category $\mathbf{D}(R\textbf{-Mod})$ one may define its **determinant**

$$\det_R P_\bullet := \bigotimes_{i\in\mathbb{Z}} \det_R H^i(P^\bullet)^{(-1)^i}.$$

0.4.1. Fact ([KM1976, p. 43, Corollary 2]). *For a distinguished triangle of perfect complexes in* $\mathbf{D}(R\textbf{-Mod})$

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

we have a canonical isomorphism

$$\det_R A^\bullet \otimes_R \det_R C^\bullet \xrightarrow{\cong} \det B^\bullet.$$

It is functorial with respect to isomorphisms of distinguished triangles: for such an isomorphism

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "\cong"', "f"] & B^\bullet \arrow[r] \arrow[d, "\cong"', "g"] & C^\bullet \arrow[r] \arrow[d, "\cong"', "h"] & A^\bullet[1] \arrow[d, "\cong"', "f[1]"] \\
A'^\bullet \arrow[r] & B'^\bullet \arrow[r] & C'^\bullet \arrow[r] & A'^\bullet[1]
\end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd}
\det_R A^\bullet \otimes \det_R C^\bullet \arrow[r, "\cong"] \arrow[d, "\det(f)\otimes\det(h)"', "\cong"] & \det_R B^\bullet \arrow[d, "\cong"', "\det(g)"] \\
\det_R A'^\bullet \otimes \det_R C'^\bullet \arrow[r, "\cong"] & \det_R B'^\bullet
\end{tikzcd}$$

Note that in particular, if we consider the direct sum of distinguished triangles

$$A^\bullet \xrightarrow{\mathrm{id}} A^\bullet \to 0 \to A^\bullet[1] \quad \text{and} \quad 0 \to B^\bullet \xrightarrow{\mathrm{id}} B^\bullet \to 0$$

then we obtain a distinguished triangle

$$A^\bullet \to A^\bullet \oplus B^\bullet \to B^\bullet \to A^\bullet[1]$$

and 0.4.1 gives us a canonical isomorphism

$$\det_R A^\bullet \otimes_R \det_R B^\bullet \xrightarrow{\cong} \det_R(A^\bullet \oplus B^\bullet).$$

0.5 Roots of unity

The m-th complex roots of unity

$$\mu_m(\mathbb{C}) := \{z \in \mathbb{C}^\times \mid z^m = 1\} = \{e^{2\pi i k/m} \mid k = 0, \ldots, m-1\}$$

form an abelian group with respect to multiplication. It also carries a natural action of the Galois group $G_\mathbb{R} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by complex conjugation, making $\mu_m(\mathbb{C})$ into a $G_\mathbb{R}$-module.

Let us fix some (standard) conventions for G-modules. We write the action of G on the left. If A and B are G-modules, then we denote by $A \otimes B$

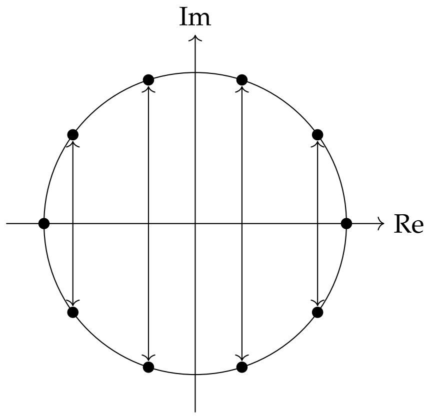
the tensor product of A and B over $\mathbb{Z}$ together with the action of G defined by

$$g(a \otimes b) := g \cdot a \otimes g \cdot b.$$

This tensor product in the category of G-modules is left adjoint to the internal Hom, which we denote by $\underline{\mathrm{Hom}}(A, B)$. The action of G on the latter is given by

$$(gf)(a) := g \cdot f(g^{-1} \cdot a)$$

for a group homomorphism $f \colon A \to B$.



The action of $G_{\mathbb{R}}$ on $\mu_{10}(\mathbb{C})$.

As an abelian group, $\mu_m(\mathbb{C})$ is non-canonically isomorphic to $\mathbb{Z}/m\mathbb{Z}$. Similarly, the group of all roots of unity $\operatorname{colim}_m \mu_m(\mathbb{C}) = \bigoplus_p \varinjlim_r \mu_{p^r}(\mathbb{C})$ is isomorphic to $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$. Now we are going to write down such isomorphisms in a canonical way, without forgetting about the action of $G_{\mathbb{R}}$. Further, we introduce a twist by n. In the setting of this text, n is a negative integer, but for the sake of completeness, let us do that for any integer n.

0.5.1. Definition (Tate twists). Let $n \in \mathbb{Z}$.

- If $n = 0$, then
$$\mu_m(\mathbb{C})^{\otimes 0} := \mathbb{Z}/m\mathbb{Z},$$
where $\mathbb{Z}/m\mathbb{Z}$ is taken with the trivial action of $G_{\mathbb{R}}$.

- If $n > 0$, then
$$\mu_m(\mathbb{C})^{\otimes n} := \underbrace{\mu_m(\mathbb{C}) \otimes \cdots \otimes \mu_m(\mathbb{C})}_{n}$$
is the n-th tensor power of $\mu_m(\mathbb{C})$ with the canonical action of $G_{\mathbb{R}}$.

- If $n < 0$, then
$$\mu_m(\mathbb{C})^{\otimes n} := \underline{\mathrm{Hom}}(\mu_m(\mathbb{C})^{\otimes(-n)}, \mathbb{Z}/m\mathbb{Z}),$$
where in this case the action of $G_\mathbb{R}$ is given by
$$\overline{f}(z) := f(\overline{z}).$$

0.5.2. Lemma. *There is a canonical isomorphism of $G_\mathbb{R}$-modules*
$$\begin{aligned} \mu_m(\mathbb{C}) &\xrightarrow{\cong} \frac{2\pi i\,\mathbb{Z}}{m\,(2\pi i)\,\mathbb{Z}}, \\ e^{2\pi i k/m} &\mapsto 2\pi i k. \end{aligned}$$

Proof. The given explicit map is pretty self-explanatory, but the reader might appreciate the fact that this comes from the snake lemma. Let us consider the following morphism of short exact sequences of $G_\mathbb{R}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & 2\pi i\,\mathbb{Z} \arrow[r] \arrow[d, "-\times m"] & \mathbb{C} \arrow[r, "z\mapsto e^z"] \arrow[d, "-\times m"] & \mathbb{C}^\times \arrow[r] \arrow[d, "(-)^m"] & 1 \\
0 \arrow[r] & 2\pi i\,\mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "z\mapsto e^z"] & \mathbb{C}^\times \arrow[r] & 1
\end{tikzcd}$$

Note that all the involved arrows are $G_\mathbb{R}$-equivariant. The map in the middle has trivial kernel and cokernel, so by the snake lemma, there is a canonical isomorphism between the kernel of the last map, which is $\mu_m(\mathbb{C})$, and the cokernel of the first map, which is $\frac{2\pi i\,\mathbb{Z}}{m\,(2\pi i)\,\mathbb{Z}}$:
$$\mu_m(\mathbb{C}) \xrightarrow{\cong} \frac{2\pi i\,\mathbb{Z}}{m\,(2\pi i)\,\mathbb{Z}}.$$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, tail] & 0 \arrow[r] \arrow[d, tail] & \mu_m(\mathbb{C}) \arrow[d, tail] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & 2\pi i\,\mathbb{Z} \arrow[r] \arrow[d, "-\times m"] & \mathbb{C} \arrow[r, "z\mapsto e^z"] \arrow[d, "-\times m"] & \mathbb{C}^\times \arrow[r] \arrow[d, "(-)^m"] & 1 \\
0 \arrow[r] & 2\pi i\,\mathbb{Z} \arrow[r] \arrow[d, two heads] & \mathbb{C} \arrow[r, "z\mapsto e^z"] \arrow[d, two heads] & \mathbb{C}^\times \arrow[r] \arrow[d, two heads] & 1 \\
 & \frac{2\pi i\,\mathbb{Z}}{m\,(2\pi i)\,\mathbb{Z}} \arrow[r] & 0 \arrow[r] & 1 \arrow[r] & 1
\end{tikzcd}$$

∎

0.5.3. Lemma. *For $n > 0$ we have a canonical isomorphism of $G_{\mathbb{R}}$-modules*

$$\mu_m(\mathbb{C})^{\otimes n} \cong \frac{(2\pi i)^n}{m\,(2\pi i)^n\,\mathbb{Z}}.$$

Proof. From the previous calculation and the canonical $G_{\mathbb{R}}$-equivariant isomorphism

$$\underbrace{(2\pi i)\,\mathbb{Z} \otimes \cdots \otimes (2\pi i)\,\mathbb{Z}}_{n} \xrightarrow{\cong} (2\pi i)^n\,\mathbb{Z},$$
$$(2\pi i)\,a_1 \otimes \cdots \otimes (2\pi i)\,a_n \mapsto (2\pi i)^n\,a_1 \cdots a_n$$

we obtain

$$\underbrace{\mu_m(\mathbb{C}) \otimes \cdots \otimes \mu_m(\mathbb{C})}_{n} \xrightarrow{\cong} \underbrace{\frac{2\pi i\,\mathbb{Z}}{m\,(2\pi i)\,\mathbb{Z}} \otimes \cdots \otimes \frac{2\pi i\,\mathbb{Z}}{m\,(2\pi i)\,\mathbb{Z}}}_{n} \cong \frac{(2\pi i)^n\,\mathbb{Z}}{m\,(2\pi i)^n\,\mathbb{Z}}.$$

■

0.5.4. Lemma. *For $n < 0$ we have a canonical isomorphism of $G_{\mathbb{R}}$-modules*

$$\mu_m(\mathbb{C})^{\otimes n} := \underline{\mathrm{Hom}}(\mu_m^{\otimes(-n)}(\mathbb{C}), \mathbb{Z}/m\mathbb{Z}) \cong \frac{(2\pi i)^n\,\mathbb{Z}}{m\,(2\pi i)^n\,\mathbb{Z}}.$$

Proof. We claim that there is a $G_{\mathbb{R}}$-equivariant isomorphism

(0.5.1)
$$\underline{\mathrm{Hom}}\left(\frac{(2\pi i)^{-n}\,\mathbb{Z}}{m\,(2\pi i)^{-n}\,\mathbb{Z}}, \mathbb{Z}/m\mathbb{Z}\right) \cong \underline{\mathrm{Hom}}\left((2\pi i)^{-n}\,\mathbb{Z},\ \mathbb{Z}/m\mathbb{Z}\right) \xrightarrow{\cong} \frac{(2\pi i)^n\,\mathbb{Z}}{m\,(2\pi i)^n\,\mathbb{Z}}.$$

Note that $-n$ got replaced with n, for the reason which will be apparent in a second. A homomorphism $f\colon (2\pi i)^{-n}\,\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is determined by the image of a generator $f((2\pi i)^{-n} \cdot 1)$, so we may define the second isomorphism in (0.5.1) by

$$\Phi\colon f \mapsto (2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1). \tag{0.5.2}$$

It is clearly an isomorphism of abelian groups, and it only remains to check that it is $G_{\mathbb{R}}$-equivariant, i.e. that for all $f\colon (2\pi i)^{-n}\,\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ holds

$$\Phi(\overline{f}) = \overline{\Phi(f)}.$$

We have indeed

$$\begin{aligned}\Phi(\overline{f}) = (2\pi i)^n \cdot \overline{f}((2\pi i)^{-n} \cdot 1) &= (2\pi i)^n \cdot f(\overline{(2\pi i)^{-n} \cdot 1}) \\ &= (-1)^n\,(2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1)\end{aligned}$$

and

$$\overline{\Phi(f)} = \overline{(2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1)} = (-1)^n\,(2\pi i)^n \cdot f((2\pi i)^{-n} \cdot 1).$$

■

0.5.5. Lemma. *The $G_{\mathbb{R}}$-module of all roots of unity twisted by n is canonically isomorphic to the $G_{\mathbb{R}}$-module $\frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}$:*

$$\operatorname*{colim}_m \mu_m(\mathbb{C})^{\otimes n} := \bigoplus_p \varinjlim_r \mu_{p^r}(\mathbb{C})^{\otimes n} \cong \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}.$$

Proof. Using the previous calculations and observing that the transition morphisms in the colimit are $G_{\mathbb{R}}$-equivariant,

$$\bigoplus_p \varinjlim_r \mu_{p^r}(\mathbb{C})^{\otimes n} \cong \bigoplus_p \varinjlim_r \frac{(2\pi i)^n\,\mathbb{Z}}{p^r\,(2\pi i)^n\,\mathbb{Z}} \cong \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}.$$

■

Somewhat abusively, from now on we will write simply "$(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}$" for $\frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}}$.

0.6 *G*-equivariant sheaves

G-equivariant sheaves on topological spaces are discussed in Grothendieck's Tohoku paper [Tôhoku]:

> Nous appelerons G-faisceau sur $X = X(G)$ un faisceau (d'ensembles) A sur X, dans lequel G opère de façon compatible avec ses opérations sur X. Pour donner un sens à cette définition, on pourra par exemple considérer A comme espace étalé dans X; nous n'insisterons pas.

In this section I will give some explanation of the notion of a G-equivariant sheaf and collect certain relevant results. What follows is a rather straightforward generalization of the usual sheaf theory, so I omit some details. Probably the best way to motivate the definition is to recall the construction of the sheaf of sections of a continuous map.

0.6.1. Classical example. Let X be a topological space. Consider the category $\mathbf{Top}_{/X}$ of spaces over X where the objects are continuous maps of topological spaces $p\colon E \to X$ and the morphisms are commutative diagrams

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd} \tag{0.6.1}$$

For a topological space over X given by $p\colon E \to X$, the corresponding **sheaf of sections** is the sheaf of sets defined by

$$\mathcal{F}(U) := \mathrm{Hom}_{\mathbf{Top}_{/X}}(U, E) = \left\{ \begin{tikzcd} U \arrow[rr, "s"] \arrow[dr] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd} \right\}$$

for each open subset $U \subset X$. The restriction maps are obvious: an inclusion of open subsets $i\colon V \hookrightarrow U$ induces contravariantly

$$\mathrm{res}_{VU} := \mathrm{Hom}_{\mathbf{Top}_{/X}}(i, E)\colon \mathcal{F}(U) \to \mathcal{F}(V),$$

and the sheaf axiom is also easy to verify. A morphism over X of the form (0.6.1) gives rise to a morphism of the corresponding sheaves of sections: for each open subset $U \subset X$ we get a map

$$\phi_U\colon \mathrm{Hom}_{\mathbf{Top}_{/X}}(U, E) \to \mathrm{Hom}_{\mathbf{Top}_{/X}}(U, E'),$$

$$\begin{tikzcd} U \arrow[rr, "s"] \arrow[dr] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd} \mapsto \begin{tikzcd} U \arrow[r, "s"] \arrow[dr] & E \arrow[r, "f"] \arrow[d, "p"] & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

and for each $V \subset U$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) := \mathrm{Hom}_{\mathbf{Top}_{/X}}(U, E) \arrow[r, "\phi_U"] \arrow[d, "\mathrm{res}_{VU}"'] & \mathrm{Hom}_{\mathbf{Top}_{/X}}(U, E') =: \mathcal{F}'(U) \arrow[d, "\mathrm{res}'_{VU}"] \\ \mathcal{F}(V) := \mathrm{Hom}_{\mathbf{Top}_{/X}}(V, E) \arrow[r, "\phi_V"] & \mathrm{Hom}_{\mathbf{Top}_{/X}}(V, E') =: \mathcal{F}'(V) \end{tikzcd}$$

clearly commutes. So formation of the sheaf of sections is a functor

$$\Gamma\colon \mathbf{Top}_{/X} \to \mathbf{Sh}(X).$$

0.6.2. *G*-equivariant example. For a discrete group G, consider the category of G-spaces G-**Top** where the objects are topological spaces X with a specified action of G by homeomorphisms $\sigma_X\colon G \times X \to X$, and morphisms $f\colon X \to Y$ are continuous G-equivariant maps:

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\sigma_X"'] & G \times Y \arrow[d, "\sigma_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

For a fixed G-space X, the category G-$\mathbf{Top}_{/X}$ of **G-spaces over** X has as its objects continuous G-equivariant maps $p\colon E \to X$ and as morphisms continuous G-equivariant maps over X

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

For a G-space over X given by $p\colon E \to X$, the corresponding sheaf of sections $\mathcal{F}$ carries the following extra datum. For each open subset $U \subset X$ and each $g \in G$ there is a bijection of sets

$$\alpha_{g,U}\colon \mathcal{F}(U) \xrightarrow{\cong} \mathcal{F}(g\cdot U),$$
$$(s\colon U \to E) \mapsto \begin{pmatrix} g\cdot U & \to & E, \\ g\cdot u & \mapsto & g\cdot s(u) \end{pmatrix},$$
$$\begin{pmatrix} U & \to & E, \\ u & \mapsto & g^{-1}\cdot s(g\cdot u) \end{pmatrix} \leftarrow\!\shortmid (s\colon g\cdot U \to E).$$

Using the fact that p is G-equivariant, one checks that $\alpha_{g,U}$ indeed sends sections over U to sections over $g \cdot U$. We also see that the bijections $\alpha_{g,U}$ satisfy the following properties:

1) compatibility with restrictions: for open subsets $V \subset U$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_{g,U}", "\cong"'] \arrow[d, "\mathrm{res}_{VU}"'] & \mathcal{F}(g\cdot U) \arrow[d, "\mathrm{res}_{g\cdot V, g\cdot U}"] \\ \mathcal{F}(V) \arrow[r, "\alpha_{g,V}", "\cong"'] & \mathcal{F}(g\cdot V) \end{tikzcd}$$

commutes;

2) for the identity element $1 \in G$ and each open subset $U \subset X$ we have

$$\alpha_{1,U} = \mathrm{id}\colon \mathcal{F}(U) \to \mathcal{F}(U);$$

3) the cocycle condition: for each open subset $U \subset X$ and $g, h \in G$ the diagram

$$\begin{tikzcd} & \mathcal{F}(h\cdot U) \arrow[dr, "\alpha_{g,h\cdot U}"] & \\ \mathcal{F}(U) \arrow[ur, "\alpha_{h,U}"] \arrow[rr, "\alpha_{gh,U}"'] & & \mathcal{F}(gh\cdot U) \end{tikzcd}$$

commutes.

For a morphism of G-spaces over X

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

the corresponding morphism of sheaves of sections $\phi\colon \mathcal{F} \to \mathcal{F}'$ is easily seen to be compatible with the maps $\alpha_{g,U}$ and $\alpha'_{g,U}$: for each $U \subset X$ the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\alpha_{g,U}"'] & \mathcal{F}'(U) \arrow[d, "\alpha'_{g,U}"] \\ \mathcal{F}(g \cdot U) \arrow[r, "\phi_{g\cdot U}"] & \mathcal{F}'(g \cdot U) \end{tikzcd}$$

commutes.

Now hopefully, the last example makes the following definition look natural.

0.6.3. Definition. Let G be a discrete group and let X be a G-space. Then a **G-equivariant presheaf** (of sets) on X is a presheaf $\mathcal{F}$ equipped with bijections of sets

$$\alpha_{g,U}\colon \mathcal{F}(U) \xrightarrow{\cong} \mathcal{F}(g \cdot U)$$

for each $g \in G$ and open subset $U \subset X$ that satisfy the following axioms:

1) these bijections are compatible with restrictions:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_{g,U}", "\cong"'] \arrow[d, "\mathrm{res}_{VU}"'] & \mathcal{F}(g \cdot U) \arrow[d, "\mathrm{res}_{g\cdot V, g\cdot U}"] \\ \mathcal{F}(V) \arrow[r, "\alpha_{g,V}", "\cong"'] & \mathcal{F}(g \cdot V) \end{tikzcd}$$

2) $\alpha_{1,U} = \mathrm{id}\colon \mathcal{F}(U) \to \mathcal{F}(U)$;

3) for $g, h \in G$ the cocycle condition holds:

$$\begin{tikzcd} & \mathcal{F}(h \cdot U) \arrow[dr, "\alpha_{g,h\cdot U}"] & \\ \mathcal{F}(U) \arrow[ur, "\alpha_{h,U}"] \arrow[rr, "\alpha_{gh,U}"'] & & \mathcal{F}(gh \cdot U) \end{tikzcd}$$

A G-**equivariant sheaf** is a G-equivariant presheaf satisfying the usual sheaf axiom: for each open covering $U = \bigcup_i U_i$ we have an equalizer

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

A **morphism of** G**-equivariant (pre)sheaves** $\mathcal{F} \to \mathcal{F}'$ is a morphism of (pre)sheaves which is compatible with the maps $\alpha_{g,U}$:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\alpha_{g,U}"'] & \mathcal{F}'(U) \arrow[d, "\alpha'_{g,U}"] \\
\mathcal{F}(g \cdot U) \arrow[r, "\phi_{g \cdot U}"] & \mathcal{F}'(g \cdot U)
\end{tikzcd}$$

We denote the category of G-equivariant presheaves (resp. sheaves) on X by $\mathbf{PSh}(G, X)$ (resp. $\mathbf{Sh}(G, X)$).

We may summarize 0.6.2 by saying that taking the sheaf of sections is a functor

$$\Gamma\colon G\text{-}\mathbf{Top}_{/X} \to \mathbf{Sh}(G, X).$$

It commutes with the forgetful functors:

$$\begin{tikzcd}
G\text{-}\mathbf{Top}_{/X} \arrow[r, "\Gamma"] \arrow[d] & \mathbf{Sh}(G, X) \arrow[d] \\
\mathbf{Top}_{/X} \arrow[r, "\Gamma"] & \mathbf{Sh}(X)
\end{tikzcd}$$

0.6.4. Remark. Despite the extra datum coming from the action of G, the category $\mathbf{Sh}(G, X)$ is still a Grothendieck topos. This can be deduced from Giraud's characterization of Grothendieck toposes [SGA 4, Exposé IV, 1.2] (see e.g. [MLM1994, Appendix] for details). However, the underlying Grothendieck site is not obvious.

0.6.5. Observation. *The global sections $\mathcal{F}(X)$ of a G-equivariant (pre)sheaf is a G-set with the action of G given by*

$$\alpha_{g,X}\colon \mathcal{F}(X) \xrightarrow{\cong} \mathcal{F}(g \cdot X) = \mathcal{F}(X).$$

Taking the global sections is a functor

$$\mathbf{PSh}(G, X) \to G\text{-}\mathbf{Set}.$$

Proof. The axioms $\alpha_{1,X} = \mathrm{id}$ and $\alpha_{gh,X} = \alpha_{g,h\cdot X} \circ \alpha_{h,X}$ correspond to the axioms of a group action. ■

0.6.6. Example. Let $\mathcal{F}$ be the sheaf of sections of a G-space over X given by $p\colon E \to X$. Then the action of $g \in G$ on $\mathcal{F}(X)$ sends a global section $s\colon X \to E$ to the global section

$$\begin{aligned} &X \to E, \\ &x \mapsto g \cdot s(g^{-1} \cdot x). \end{aligned}$$

(see the formula for $\alpha_{g,U}$ in 0.6.2). ▲

0.6.7. Definition. Let S be a G-set. For a G-space X, consider the presheaf S_X defined by $S_X(U) = S$ for each open subset $U \subset X$ with the identity restriction maps. The morphisms

$$\begin{aligned} \alpha_{g,U} = \sigma_g\colon S_X(U) &\to S_X(g \cdot U), \\ x &\mapsto g \cdot x. \end{aligned}$$

define a structure of a G-equivariant presheaf on S_X, called the **constant G-equivariant presheaf** associated to S.

0.6.8. Observation. *Formation of the constant G-equivariant presheaf is a functor*

$$G\text{-}\mathbf{Set} \to \mathbf{PSh}(G,X),$$

which is left adjoint to the global section functor:

$$\mathrm{Hom}_{\mathbf{PSh}(G,X)}(S_X, \mathcal{P}) \cong \mathrm{Hom}_{G\text{-}\mathbf{Set}}(S, \mathcal{P}(X)).$$

Proof. A morphism of G-equivariant presheaves $S_X \to \mathcal{P}$ is given by a collection of maps $\phi_U\colon S \to \mathcal{P}(U)$ that are compatible with the restriction maps and the G-equivariant structure morphisms:

$$\begin{tikzcd} S \arrow[r, "\phi_X"] \arrow[dr, "\phi_U"] \arrow[ddr, "\phi_V"'] & \mathcal{P}(X) \arrow[d, "\mathrm{res}_{UX}"] \arrow[dd, bend left=60, "\mathrm{res}_{VX}"] \\ & \mathcal{P}(U) \arrow[d, "\mathrm{res}_{VU}"] \\ & \mathcal{P}(V) \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\phi_U"] \arrow[d, "\sigma_g"'] & \mathcal{P}(U) \arrow[d, "\alpha_{g,U}"] \\ S \arrow[r, "\phi_{g\cdot U}"] & \mathcal{P}(g \cdot U) \end{tikzcd}$$

From the first diagram we see that $\phi_U = \mathrm{res}_{UX} \circ \phi_X$, so that the map $\phi_X\colon S \to \mathcal{P}(X)$ defines the rest, and from the second diagram we see that it is G-equivariant. This shows that the bijection in question is given by

$$\{\phi_U\} \mapsto \phi_X,$$
$$\{\phi_U := \mathrm{res}_{UX} \circ \phi\} \leftarrow\!\shortmid \phi.$$

■

Alternative definition via G-equivariant espaces étalés

One says that a continuous map $p\colon E \to X$ is **étale*** if it is a local on the source homeomorphism (for each $e \in E$ there exists an open neighborhood $V \ni p$ such that $p(V)$ is open in X and $p|_V : V \to p(V)$ is a homeomorphism). We have a full subcategory

$$G\text{-}\mathbf{\acute{E}t}_{/X} \subset G\text{-}\mathbf{Top}_{/X}$$

formed by G-spaces that are étale over X. We note that if p and p' are étale and we have a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

then f is étale as well, so that the morphisms in $G\text{-}\mathbf{\acute{E}t}_{/X}$ are automatically étale. The importance of étale spaces over X is explained by the following well-known result, which we state G-equivariantly.

0.6.9. Proposition. *Let $\mathcal{F}$ be a G-equivariant presheaf on X. Consider the disjoint union of stalks*

$$\coprod_{x\in X} \mathcal{F}_x, \quad \mathcal{F}_x := \varinjlim_{U\ni x} \mathcal{F}(U).$$

It carries a natural action of G. For each section $s \in \mathcal{F}(U)$ such that $U \ni x$, denote by $s_x \in \mathcal{F}_x$ the corresponding germ at x. This defines a map between sets (which we again denote by s)

$$s\colon U \to \coprod_{x\in X} \mathcal{F}_x,$$
$$x \mapsto s_x.$$

Consider now the topology on $\coprod_{x\in X} \mathcal{F}_x$ generated by $s(U)$ for all open subsets $U \subset X$ and all $s \in \mathcal{F}(U)$. Then the action of G is continuous with respect to this

*This is in fact the topological counterpart of étale morphisms of schemes.

topology, and the natural projection

$$p\colon \coprod_{x\in X} \mathcal{F}_x \to X,$$
$$\mathcal{F}_x \ni s_x \mapsto x.$$

is an étale G-equivariant map.

Proof. This is a well-known, basic result (see e.g. [MLM1994, Chapter II]); one just has to check the G-equivariance. ■

This leads to an equivalent definition of G-equivariant sheaves.

0.6.10. Alternative definition. Let G be a group and X be a G-space. Then a G-**equivariant sheaf** on X is an étale G-space over X

$$p\colon E \to X,$$

and a morphism of G-equivariant sheaves is a morphism over X

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

0.6.11. Remark. Note that the above definition looks more natural than 0.6.3. It also generalizes to the case a topological group G acting on E and X continuously. This is not possible in 0.6.3, because there we consider only how each separate element $g \in G$ acts on X.

0.6.12. Example. In these terms, it is easier to describe equivariant sheafification and what a constant sheaf is. If S is a G-set and X is a G-space, we may endow S with the discrete topology and consider the G-space $S \times X$ with the component-wise action of G (which is the product in the category of G-spaces). Then the projection $S \times X \to X$ is an étale G-equivariant map, so it corresponds to some G-equivariant sheaf. We call it the **constant G-equivariant sheaf** associated to S. This construction is obviously functorial: a G-equivariant map $S \to S'$ induces a morphism in G-$\mathbf{\acute{E}t}_{/X}$

$$\begin{tikzcd} S \times X \arrow[rr] \arrow[dr] & & S' \times X \arrow[dl] \\ & X & \end{tikzcd}$$

▲

Abelian G-equivariant sheaves and their cohomology

0.6.13. Proposition. *Let X be a G-space. Consider the category $\mathbf{Sh}(G,X)^{\mathbf{Ab}}$ of G-equivariant sheaves of abelian groups on X (defined, for instance, as abelian group objects in the category of G-equivariant sheaves of sets). It is an abelian category with enough injectives.*

Proof. The usual argument of Grothendieck works: any abelian category which satisfies the axiom AB5) and has generators has enough injectives [Tôhoku, Ch. I, 1.10]. This is the case for $\mathbf{Sh}(G,X)^{\mathbf{Ab}}$ (as for the generators, see [MLM1994, Appendix]). ■

0.6.14. Example. Let A be a G-set (resp. G-module). Then the associated constant sheaf $\underline{A}$ has a canonical G-equivariant abelian sheaf structure. ▲

0.6.15. Example. Consider some topological space with an action of the Galois group $G_{\mathbb{R}} := \operatorname{Gal}(\mathbb{C}/\mathbb{R})$; for instance, the set of complex points of a scheme $X(\mathbb{C})$ equipped with the analytic topology. Then the complex m-th roots of unity $\mu_m(\mathbb{C})$ (reviewed above in §0.5) give us a constant $G_{\mathbb{R}}$-equivariant sheaf on $X(\mathbb{C})$. This is the only example we will be interested in. ▲

0.6.16. Definition. The equivariant global section functor

$$\begin{aligned} \Gamma(G,X,-)\colon \mathbf{Sh}(G,X)^{\mathbf{Ab}} &\to \mathbf{Ab}, \\ \mathcal{F} &\rightsquigarrow \mathcal{F}(X)^G \end{aligned}$$

is left exact. Here the global sections

$$\mathcal{F}(X) := \{s\colon X \to \text{Ét}(\mathcal{F}) \mid \pi \circ s = \mathrm{id}_X\}$$

come with an action of G by

$$(g\cdot s)(x) := g\cdot s(g^{-1}\cdot x).$$

(Note that in general, $\mathcal{F}(U)$ carries such an action of G, whenever $U \subset X$ is closed under the action of G.) The fixed points of this action are precisely the G-equivariant sections, i.e. sections that satisfy $s(g\cdot x) = g\cdot (s(x))$. The right derived functors of $\Gamma(G,X,-)$ are by definition $R\Gamma(G,X,\mathcal{F})$.

This is related to the usual sheaf cohomology by

$$R\Gamma(G,X,\mathcal{F}) \cong R\Gamma(G, R\Gamma(X,\mathcal{F})), \tag{0.6.2}$$

where the right hand side is the group cohomology. Indeed, $\Gamma(G, X, -)$ is a composition of two left exact functors: the usual global section functor and the fixed points functor

$$\mathbf{Sh}(G, X)^{\mathbf{Ab}} \xrightarrow{\mathcal{F} \rightsquigarrow \mathcal{F}(X)} G\text{-}\mathbf{Mod} \xrightarrow{A \rightsquigarrow A^G} \mathbf{Ab}$$

and (0.6.2) are the derived functors of a composition of functors (this is known as the **Grothendieck spectral sequence**; see e.g. [Wei1994, §10.8]). On the level of cohomology, we have a spectral sequence

$$E_2^{pq} = H^p(G, H^q(X, \mathcal{F})) \Longrightarrow H^{p+q}(G, X, \mathcal{F}).$$

0.7 From étale to analytic sheaves (the morphism α^*)

The canonical reference for comparison between étale and singular cohomology is [SGA 4, Exposé XI, §4], so let us to borrow some definitions and notation from there. Let X be an arithmetic scheme (separated, of finite type over $\operatorname{Spec} \mathbb{Z}$).

1. The base change from $\operatorname{Spec} \mathbb{Z}$ to $\operatorname{Spec} \mathbb{C}$

$$\begin{tikzcd} X_{\mathbb{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

   gives us a morphism of sites

$$\gamma \colon X_{\mathbb{C},\acute{e}t} \to X_{\acute{e}t}.$$

2. We denote by $X(\mathbb{C})$ the set of complex points of X equipped with the usual analytic topology.

   Let X_{cl} be the site of étale maps $f \colon U \to X(\mathbb{C})$. A covering family in X_{cl} is a family of maps $\{U_i \to U\}$ such that U is the union of images of U_i. The notation "cl" comes from SGA 4 and stays for "classique".

   As the inclusion of an open subset $U \subset X(\mathbb{C})$ is trivially an étale map, we have a fully faithful functor $X(\mathbb{C}) \subset X_{cl}$, and the topology on $X(\mathbb{C})$ is induced by the topology on X_{cl}. This gives us a morphism of sites

$$\delta \colon X_{cl} \to X(\mathbb{C}),$$

   which by the well-known "comparison lemma" [SGA 4, Exposé III, Théorème 4.1] induces an equivalence of the corresponding categories of sheaves

$$\delta_* \colon \mathbf{Sh}(X_{cl}) \to \mathbf{Sh}(X(\mathbb{C})).$$

3. A morphism of schemes $f\colon X'_{\mathbb{C}} \to X_{\mathbb{C}}$ over $\operatorname{Spec}\mathbb{C}$ is étale if and only if $f(\mathbb{C})\colon X'(\mathbb{C}) \to X(\mathbb{C})$ is étale in the topological sense [SGA 1, Exposé XII, Proposition 3.1], and therefore the functor $X'_{\mathbb{C}} \rightsquigarrow X'(\mathbb{C})$ gives us a morphism of sites
$$\epsilon\colon X_{cl} \to X_{\mathbb{C},ét}.$$

We may now consider the composite functor

$$\mathbf{Sh}(X_{ét}) \xrightarrow{\gamma^*} \mathbf{Sh}(X_{\mathbb{C},ét}) \xrightarrow{\epsilon^*} \mathbf{Sh}(X_{cl}) \xrightarrow[\simeq]{\delta_*} \mathbf{Sh}(X(\mathbb{C}))$$

where γ^* is given by the base change from $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec}\mathbb{C}$, the functor ϵ^* is the comparison, and δ_* is an equivalence of categories. As we start from a scheme over $\operatorname{Spec}\mathbb{Z}$ and base change to $\operatorname{Spec}\mathbb{C}$, the resulting sheaf on $X(\mathbb{C})$ is in fact equivariant with respect to the complex conjugation, and the above composition gives us an “inverse image” functor

$$\alpha^*\colon \mathbf{Sh}(X_{ét}) \to \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C})).$$

0.8 Cohomology with compact support on $X_{ét}$ and $X(\mathbb{C})$

For any arithmetic scheme $f\colon X \to \operatorname{Spec}\mathbb{Z}$ (separated, of finite type) there exists a **Nagata compactification** $f = g \circ j$ where j is an open immersion and g is a proper morphism:

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathfrak{X} \arrow[dl, "g"] \\ & \operatorname{Spec}\mathbb{Z} & \end{tikzcd}$$

This is a result of Nagata, and a modern exposition (following Deligne) may be found in [Con2007, Con2009]. See also [SGA 4, Exposé XVII].

0.8.1. Definition. Let X be an arithmetic scheme and let $\mathcal{F}^\bullet$ be a complex of abelian torsion sheaves on $X_{ét}$. Then we define the **cohomology of $\mathcal{F}^\bullet$ with compact support** via the complex

$$R\Gamma_c(X_{ét}, \mathcal{F}^\bullet) := R\Gamma(\mathfrak{X}_{ét}, j_!\mathcal{F}^\bullet). \tag{0.8.1}$$

For torsion sheaves, this does not depend on the choice of $j\colon X \hookrightarrow \mathfrak{X}$, but here we would like to fix this choice to be able to compare j with the corresponding morphism $j(\mathbb{C})\colon X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$. Note that thanks to the **Leray spectral sequence** $R\Gamma(\mathfrak{X}_{ét}, -) \cong R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, -) \circ Rg_*$ (that is, the

Grothendieck spectral sequence coming from $\Gamma(\mathfrak{X}_{ét}, -) = \Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, -) \circ g_*)$, we have

$$R\Gamma_c(X_{ét}, \mathcal{F}^\bullet) \cong R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, Rf_!\mathcal{F}^\bullet), \tag{0.8.2}$$

where by definition

$$Rf_!\mathcal{F}^\bullet := Rg_* j_! \mathcal{F}^\bullet$$

(this is just a piece of notation, standard and quite unfortunate; "$Rf_!$" does not mean that we are deriving $f_!$).

The formulas (0.8.1) and (0.8.2) give two equivalent definitions. We are going to use (0.8.2) in the next section to introduce a slightly different version of cohomology with compact support, denoted by $R\widehat{\Gamma}_c(X_{ét}, \mathcal{F}^\bullet)$, which is needed for arithmetic duality theorems. In this section, we need to use (0.8.1) to define cohomology with compact support on $X(\mathbb{C})$, in a way that allows us to compare it with cohomology with compact support on $X_{ét}$.

0.8.2. Definition. If $j\colon X \hookrightarrow \mathfrak{X}$ is a Nagata compactification, then we have the corresponding open immersion

$$j(\mathbb{C})\colon X(\mathbb{C}) \to \mathfrak{X}(\mathbb{C}),$$

and for a sheaf $\mathcal{F}$ on $X(\mathbb{C})$ we define

$$\Gamma_c(X(\mathbb{C}), \mathcal{F}) := \Gamma(\mathfrak{X}(\mathbb{C}), j(\mathbb{C})_!\mathcal{F}).$$

Similarly, for a $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$ we define

$$\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \mathcal{F}) := \Gamma(G_\mathbb{R}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_!\mathcal{F}).$$

0.8.3. Proposition. *Let $\mathcal{F}$ be a sheaf on $X_{ét}$.*

1) *There exists a morphism*

$$\Gamma(X_{ét}, \mathcal{F}) \to \Gamma(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}),$$

which is natural in the sense that every morphism of sheaves $\mathcal{F} \to \mathcal{G}$ gives a commutative diagram

$$\begin{tikzcd}
\Gamma(X_{ét}, \mathcal{F}) \arrow[r] \arrow[d] & \Gamma(X_{ét}, \mathcal{G}) \arrow[d] \\
\Gamma(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}) \arrow[r] & \Gamma(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{G})
\end{tikzcd}$$

2) *Similarly for cohomology with compact support, there is a natural morphism*

$$\Gamma_c(X_{ét}, \mathcal{F}) \to \Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}).$$

The same holds for abelian sheaves on $X_{ét}$.

Proof. This is standard and follows from the functoriality of α^*, but it is easier to recall the construction than to find the relevant point in SGA 4. The morphism in 1) is given by

$$\begin{aligned}\Gamma(X_{ét}, \mathcal{F}) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Sh}(X_{ét})}(\underline{\{*\}}, \mathcal{F}) \to \mathrm{Hom}_{\mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))}(\alpha^* \underline{\{*\}}, \alpha^* \mathcal{F}) \\ &\xrightarrow{\cong} \mathrm{Hom}_{\mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C}))}(\underline{\{*\}}, \alpha^* \mathcal{F}) \xrightarrow{\cong} \Gamma(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}).\end{aligned}$$

For abelian sheaves, in the above formula one has to replace the constant sheaf $\underline{\{*\}}$ with $\underline{\mathbb{Z}}$. The naturality is easily seen from the above definition.

In 2), if $j\colon X \hookrightarrow \mathfrak{X}$ is Nagata compactification, then we have the corresponding compactification $j(\mathbb{C})\colon X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$. The extension by zero morphism $j(\mathbb{C})_!\colon \mathbf{Sh}(X(\mathbb{C})) \to \mathbf{Sh}(\mathfrak{X}(\mathbb{C}))$ restricts to the subcategory of $G_{\mathbb{R}}$-equivariant sheaves: if $\mathcal{F}$ is a $G_{\mathbb{R}}$-equivariant sheaf on $X(\mathbb{C})$, then $j(\mathbb{C})_! \mathcal{F}$ is a $G_{\mathbb{R}}$-equivariant sheaf on $\mathfrak{X}(\mathbb{C})$ (this is evident from the definition of equivariant sheaves as equivariant espaces étalés). It should be clear from the definition of α^* that there is a commutative diagram

$$\begin{tikzcd} \mathbf{Sh}(X_{ét}) \arrow[r, "\alpha^*"] \arrow[d, "j_!"'] & \mathbf{Sh}(G_{\mathbb{R}}, X(\mathbb{C})) \arrow[d, "j(\mathbb{C})_!"] \\ \mathbf{Sh}(\mathfrak{X}_{ét}) \arrow[r, "\alpha^*_{\mathfrak{X}}"'] & \mathbf{Sh}(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C})) \end{tikzcd}$$

(For instance, note that this diagram commutes for representable étale sheaves, and then every étale sheaf is a colimit of representable sheaves, and α^*, $j_!$, $\alpha^*_{\mathfrak{X}}$, $j(\mathbb{C})_!$ preserve colimits, as left adjoints.)

Now the morphism in question is now given by

$$\begin{aligned}\Gamma_c(X_{ét}, \mathcal{F}) := \Gamma(\mathfrak{X}_{ét}, j_! \mathcal{F}) &\to \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha^*_{\mathfrak{X}} j_! \mathcal{F}) \\ &= \Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), j(\mathbb{C})_!\, \alpha^* \mathcal{F}) =: \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F}).\end{aligned}$$

■

Finally, we will need the fact that the morphisms

$$\Gamma_c(X_{ét}, \mathcal{F}) \to \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^* \mathcal{F})$$

are compatible with the distinguished triangles associated to open-closed decompositions. To check this compatibility, let us recall how such triangles arise. If we have an open subscheme $U \subset X$ and its closed complement $Z := X \setminus U$:

$$Z \xhookrightarrow{i_Z} X \xhookleftarrow{j_U} U$$

then there are the following six functors between the corresponding categories of abelian sheaves:

$$\begin{tikzcd} \mathbf{Sh}(Z_{ét})^{\mathbf{Ab}} \arrow[r, "i_{Z*}" description] & \mathbf{Sh}(X_{ét})^{\mathbf{Ab}} \arrow[l, bend right, "i_Z^*"'] \arrow[l, bend left, "i_Z^!"] \arrow[r, "j_U^*" description] & \mathbf{Sh}(U_{ét})^{\mathbf{Ab}} \arrow[l, bend right, "j_{U!}"'] \arrow[l, bend left, "j_{U*}"] \end{tikzcd}$$

(see e.g. [SGA 4, Exposé 4, §14]). Here each arrow is left adjoint to the arrow depicted below it. For an abelian sheaf $\mathcal{F}$ on $X_{ét}$, there is a natural short exact sequence

$$0 \to j_{U!} j_U^* \mathcal{F} \to \mathcal{F} \to i_{Z*} i_Z^* \mathcal{F} \to 0$$

(naturality means that the two arrows are counit and unit of the corresponding adjunctions). Now if $j\colon X \to \mathfrak{X}$ is a Nagata compactification, then the above short exact sequence gives us a short exact sequence of abelian sheaves on $\mathfrak{X}_{\text{ét}}$ (the functor $j_!$ is exact):

$$0 \to j_! j_{U!} j_U^* \mathcal{F} \to j_! \mathcal{F} \to j_! i_{Z*} i_Z^* \mathcal{F} \to 0$$

and finally, this gives the distinguished triangle

$$R\Gamma(\mathfrak{X}_{\text{ét}}, j_! j_{U!} j_U^* \mathcal{F}) \to R\Gamma(\mathfrak{X}_{\text{ét}}, j_! \mathcal{F}) \to R\Gamma(\mathfrak{X}_{\text{ét}}, j_! i_{Z*} i_Z^* \mathcal{F}) \to R\Gamma(\mathfrak{X}_{\text{ét}}, j_! j_{U!} j_U^* \mathcal{F})[1]$$

which we may write as

$$R\Gamma_c(U_{\text{ét}}, \mathcal{F}|_U) \to R\Gamma_c(X_{\text{ét}}, \mathcal{F}) \to R\Gamma_c(Z_{\text{ét}}, \mathcal{F}|_Z) \to R\Gamma_c(U_{\text{ét}}, \mathcal{F}|_U)[1]$$

For ($G_\mathbb{R}$-equivariant) sheaves on $X(\mathbb{C})$, such triangles arise in the same manner.

0.8.4. Proposition. *For an open-closed decomposition*

$$Z \xhookrightarrow{i_Z} X \xhookleftarrow{j_U} U$$

the morphism α^* gives a morphism of distinguished triangles

$$\begin{tikzcd} R\Gamma_c(U_{ét}, \mathcal{F}|_U) \arrow[r] \arrow[d] & R\Gamma_c(G_\mathbb{R}, U(\mathbb{C}), \alpha^*\mathcal{F}|_{U(\mathbb{C})}) \arrow[d] \\ R\Gamma_c(X_{ét}, \mathcal{F}) \arrow[r] \arrow[d] & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), \alpha^*\mathcal{F}) \arrow[d] \\ R\Gamma_c(Z_{ét}, \mathcal{F}|_Z) \arrow[r] \arrow[d] & R\Gamma_c(G_\mathbb{R}, Z(\mathbb{C}), \alpha^*\mathcal{F}|_{Z(\mathbb{C})}) \arrow[d] \\ R\Gamma_c(U_{ét}, \mathcal{F}|_U)[1] \arrow[r] & R\Gamma_c(G_\mathbb{R}, U(\mathbb{C}), \alpha^*\mathcal{F}|_{U(\mathbb{C})})[1] \end{tikzcd} \tag{0.8.3}$$

Proof. Since α^* is essentially the inverse image functor associated to a continuous morphism of sites, it is exact, and therefore the short exact sequence on $\mathfrak{X}_{\text{ét}}$

$$0 \to j_! j_{U!} j_U^* \mathcal{F} \to j_! \mathcal{F} \to j_! i_{Z*} i_Z^* \mathcal{F} \to 0$$

gives a short exact sequence of equivariant sheaves on $\mathfrak{X}(\mathbb{C})$

$$0 \to \alpha_{\mathfrak{X}}^* j_! j_{U!} j_U^* \mathcal{F} \to \alpha_{\mathfrak{X}}^* j_! \mathcal{F} \to \alpha_{\mathfrak{X}}^* j_! i_{Z*} i_Z^* \mathcal{F} \to 0$$

This gives us the corresponding morphism of triangles

$$\begin{array}{ccc}
R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}|_U) & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! j_{U!} j_U^* \mathcal{F}) \\
\downarrow & & \downarrow \\
R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}) & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! \mathcal{F}) \\
\downarrow & & \downarrow \\
R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}|_Z) & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! i_{Z*} i_Z^* \mathcal{F}) \\
\downarrow & & \downarrow \\
R\Gamma(\mathfrak{X}_{\text{ét}}, \mathcal{F}|_U)[1] & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), \alpha_{\mathfrak{X}}^* j_! j_{U!} j_U^* \mathcal{F})[1]
\end{array}$$

Then it is possible to verify that the right triangle coincides with the one obtained from the short exact sequence of $G_{\mathbb{R}}$-equivariant sheaves on $X(\mathbb{C})$

$$0 \to j_U(\mathbb{C})_! j_U(\mathbb{C})^* \alpha^* \mathcal{F} \to \alpha^* \mathcal{F} \to i_Z(\mathbb{C})_* i_Z(\mathbb{C})^* \alpha^* \mathcal{F} \to 0$$

by applying $j(\mathbb{C})_! \colon X(\mathbb{C}) \hookrightarrow \mathfrak{X}(\mathbb{C})$ and $R\Gamma(G_{\mathbb{R}}, \mathfrak{X}(\mathbb{C}), -)$, i.e. the right column in (0.8.3). ■

0.9 Étale cohomology with compact support à la Milne

Let us first recall the definition of Tate cohomology (see e.g. [Bro1994, Chapter VI]). Let G be a finite group. Then the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a resolution by *finitely generated* free $\mathbb{Z}G$-modules

(0.9.1) $$(P_\bullet \twoheadrightarrow \mathbb{Z})\colon \quad \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

(for instance, the bar-resolution will do). The group cohomology of G with coefficients in a G-module A is the cohomology of the complex of abelian groups

$$R\Gamma(G, A) := \operatorname{Hom}_{\mathbb{Z}G}(P_\bullet, A),$$

i.e.,

$$H^i(G,A) = H^i(R\Gamma(G,A)).$$

If we dualize (0.9.1) by applying the functor $(-)^\vee := \underline{\mathrm{Hom}}(-,\mathbb{Z}G)$, then $P_i^\vee$ are also finitely generated free $\mathbb{Z}G$-modules, and we obtain a "backwards resolution", which is an acyclic complex

$$(\mathbb{Z} \rightarrowtail P_\bullet^\vee)\colon \quad 0 \to \mathbb{Z} \to P_0^\vee \to P_1^\vee \to P_2^\vee \to \cdots \tag{0.9.2}$$

We may splice together (0.9.1) and (0.9.2) to obtain a so-called **complete resolution** (with homological numbering)

$$\widehat{P}_\bullet\colon \qquad \cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

where $P_i := P^\vee_{-i-1}$ for $i < 0$, and the morphism $P_0 \to P_{-1}$ is given by the composition of $P_0 \twoheadrightarrow \mathbb{Z}$ and $\mathbb{Z} \rightarrowtail P_0^\vee$. Then the **Tate cohomology** of G with coefficients in a G-module A is given by the cohomology of the complex

$$R\widehat{\Gamma}(G,A) := \mathrm{Hom}_{\mathbb{Z}G}(\widehat{P}_\bullet, A);$$

that is,

$$\widehat{H}^i(G,A) := H^i(R\widehat{\Gamma}(G,A)).$$

This corresponds to the usual cohomology in positive degrees $i > 0$ and homology in degrees $i < -1$:

$$\widehat{H}^i(G,A) = \begin{cases} H^i(G,A), & i > 0, \\ H_{-i-1}(G,A), & i < -1, \end{cases}$$

while the groups $\widehat{H}^{-1}(G,A)$ and $\widehat{H}^0(G,A)$ are given by the exact sequence

$$0 \to \widehat{H}^{-1}(G,A) \to H_0(G,A) \xrightarrow{\overline{N}} H^0(G,A) \to \widehat{H}^0(G,A) \to 0$$

where $\overline{N}\colon H_0(G,A) \to H^0(G,A)$ is the norm map induced by $N := \sum_{g\in G} g$.

Slightly more generally, if $A^\bullet$ is a bounded below (cohomological) complex of G-modules, we obtain a *double complex* of abelian groups $\mathrm{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)$ (resp. $\mathrm{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)$), and it makes sense to define the corresponding **group hypercohomology** (resp. **Tate hypercohomology**) by the complex

$$R\Gamma(G,A^\bullet) := \mathrm{Tot}^\oplus(\mathrm{Hom}^{\bullet\bullet}(P_\bullet, A^\bullet)),$$
$$R\widehat{\Gamma}(G,A^\bullet) := \mathrm{Tot}^\oplus(\mathrm{Hom}^{\bullet\bullet}(\widehat{P}_\bullet, A^\bullet)).$$

Note that there is an obvious morphism of complexes $\widehat{P}_\bullet \to P_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\mathrm{id}"] & P_1 \arrow[r] \arrow[d, "\mathrm{id}"] & P_0 \arrow[r] \arrow[d, "\mathrm{id}"] & P_{-1} \arrow[r] \arrow[d] & P_{-2} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which after applying the contravariant functor $\operatorname{Tot}^{\oplus} \operatorname{Hom}^{\bullet\bullet}(-, A^\bullet)$ gives a morphism from the usual cohomology to Tate cohomology:

$$R\Gamma(G, A^\bullet) \to R\widehat{\Gamma}(G, A^\bullet). \tag{0.9.3}$$

0.9.1. Example. If G is a finite cyclic group of order m generated by an element t, then it admits a periodic free resolution

$$\cdots \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where

$$N := \sum_{g \in G} g = 1 + t + t^2 + \cdots + t^{m-1}$$

is the norm element, and

$$\epsilon \colon \sum_{g \in G} n_g\, g \mapsto \sum_{g \in G} n_g$$

is the augmentation morphism. If we dualize the above resolution, we get the acyclic complex

$$0 \to \mathbb{Z} \xrightarrow{\epsilon^\vee} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \to \cdots$$

It is easily seen that the morphism $\epsilon^\vee$ is given by $1 \mapsto N$, and the composition $\epsilon^\vee \circ \epsilon$ is the action by N on $\mathbb{Z}G$. The corresponding complete resolution is

(0.9.4)

$$\widehat{P}_\bullet\colon \quad \cdots \to \underset{3}{\mathbb{Z}G} \xrightarrow{t-1} \underset{2}{\mathbb{Z}G} \xrightarrow{N} \underset{1}{\mathbb{Z}G} \xrightarrow{t-1} \underset{0}{\mathbb{Z}G} \xrightarrow{N} \underset{-1}{\mathbb{Z}G} \xrightarrow{t-1} \underset{-2}{\mathbb{Z}G} \xrightarrow{N} \underset{-3}{\mathbb{Z}G} \to \cdots$$

After applying $\underline{\operatorname{Hom}}_{\mathbb{Z}G}(-, A)$, we obtain a periodic cohomological complex

$$\cdots \to \underset{-3}{A} \xrightarrow{N} \underset{-2}{A} \xrightarrow{t-1} \underset{-1}{A} \xrightarrow{N} \underset{0}{A} \xrightarrow{t-1} \underset{1}{A} \xrightarrow{N} \underset{2}{A} \xrightarrow{t-1} \underset{3}{A} \to \cdots$$

So that

$$\widehat{H}^i(G, A) \cong \begin{cases} A^G / NA, & i \text{ even}, \\ \{a \in A \mid N \cdot a = 0\} / (t-1)\, A, & i \text{ odd}. \end{cases}$$

Recall that if G is any finite group, then its homology $H_i(G, A)$ and cohomology $H^i(G, A)$ groups are annihilated by multiplication by $\#G$ for $i > 0$. In fact, this follows from a stronger result: if $P_\bullet \twoheadrightarrow \mathbb{Z}$ is the bar resolution, then the morphism

$$\begin{aligned} \text{“}\#G\text{''}\colon P_\bullet &\to P_\bullet, \\ (\#G - N)\colon P_0 &\to P_0, \\ \#G\colon P_i &\to P_i \quad \text{for } i > 1, \end{aligned}$$

which induces multiplication by $\#G$ on $H_i(G, A)$ and $H^i(G, A)$ for $i > 0$, is null-homotopic—see e.g. [Wei1994, Theorem 6.5.8]. In our case, when G is cyclic of order m, for the 2-periodic complete resolution (0.9.4), it is easy to see that the multiplication by m on $\widehat{P}_\bullet$ is null-homotopic. Indeed, such a null homotopy is also 2-periodic, and should be given by a family of morphisms

$$h^0\colon \mathbb{Z}G \to \mathbb{Z}G, \quad h^1\colon \mathbb{Z}G \to \mathbb{Z}G$$

Satisfying

$$h^0 \circ (t-1) + N \circ h^1 = m, \quad h^1 \circ N + (t-1) \circ h^0 = m. \tag{0.9.5}$$

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}G \arrow[r] & \mathbb{Z}G \arrow[r, "t-1"] \arrow[dl, "h^1"'] \arrow[d, "\#G"] & \mathbb{Z}G \arrow[r, "N"] \arrow[dl, "h^0"'] \arrow[d, "\#G"] & \mathbb{Z}G \arrow[r] \arrow[dl, "h^1"'] & \cdots \\ \cdots \arrow[r] & \mathbb{Z}G \arrow[r, "N"] & \mathbb{Z}G \arrow[r, "t-1"] & \mathbb{Z}G \arrow[r] & \mathbb{Z}G \arrow[r] & \cdots \end{tikzcd}$$

Let h^0 be the multiplication by $-x \in \mathbb{Z}G$, where

$$x := (m-1) + (m-2)\,t + (m-3)\,t^2 + \cdots + t^{m-2},$$

and let h^1 be the identity map. Then

$$\begin{aligned} x \cdot (t-1) &= (m-1)\,t + (m-2)\,t^2 + (m-3)\,t^3 + \cdots + t^{m-1} \\ &\qquad - (m-1) - (m-2)\,t - (m-3)\,t^2 - \cdots - t^{m-1} \\ &= -m + 1 + t + t^2 + \cdots + t^{m-1} = -m + N, \end{aligned}$$

so that

$$(-x) \cdot (t-1) + N = m,$$

which means that (0.9.5) is satisfied. This implies that the groups $\widehat{H}^i(G, A)$ are annihilated by m for all i, and in general, for any bounded below complex of G-modules $A^\bullet$, the groups $\widehat{H}^i(G, A^\bullet)$ are annihilated by m. The latter is evident from our argument and not so evident from the spectral sequence

$$E_2^{pq} = \widehat{H}^q(G, H^p(A^\bullet)) \Longrightarrow \widehat{H}^{p+q}(G, A^\bullet).$$

▲

We use Tate cohomology to define étale cohomology with compact support à la Milne [Mil2006, §II.2]. If $\mathcal{F}^\bullet$ is a bounded below complex of abelian sheaves on $\operatorname{Spec}\mathbb{Z}_{ét}$, then by definition, $R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet)$ is the complex sitting in the distinguished triangle

$$R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet) \to R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet) \to R\widehat{\Gamma}(G_\mathbb{R}, \mathcal{F}^\bullet_\mathbb{C}) \to R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet)[1]$$

where $R\widehat{\Gamma}(G_\mathbb{R}, \mathcal{F}^\bullet_\mathbb{C})$ is the Tate cohomology defined above, and $\mathcal{F}^\bullet_\mathbb{C}$ is the complex of $G_\mathbb{R}$-modules obtained by taking the stalks at $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$. The morphism $R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet) \to R\widehat{\Gamma}(G_\mathbb{R}, \mathcal{F}^\bullet_\mathbb{C})$ arises as follows.

The canonical morphism $v\colon \operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{Z}$ induces a morphism

$$R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet) \to R\Gamma(\operatorname{Spec}\mathbb{R}_{ét}, v^*\mathcal{F}^\bullet), \tag{0.9.6}$$

and the cohomology on $\operatorname{Spec}\mathbb{R}_{ét}$ corresponds to the cohomology of the Galois group $G_\mathbb{R}$: specifically, we have an equivalence of categories

$$\begin{gathered}\mathbf{Sh}(\operatorname{Spec}\mathbb{R}_{ét})^{\mathbf{Ab}} \xrightarrow{\simeq} G_\mathbb{R}\text{-}\mathbf{Mod},\\ \mathcal{F} \rightsquigarrow \mathcal{F}_\mathbb{C}\end{gathered}$$

—see [SGA 4, Exposé VII, 2.3]. We may thus see (0.9.6) as a morphism*

$$R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet) \to R\Gamma(G_\mathbb{R}, \mathcal{F}^\bullet_\mathbb{C}),$$

which we may compose with the morphism (0.9.3) to the Tate cohomology $R\widehat{\Gamma}(G_\mathbb{R}, \mathcal{F}^\bullet_\mathbb{C})$.

The notation "$R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, -)$" is not standard; for instance, Geisser in [Gei2010] writes "$R\Gamma_c(\operatorname{Spec}\mathbb{Z}_{ét}, -)$" for the same thing. We will use the notation "$R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, -)$" to avoid any confusion with the usual étale cohomology with compact support, as defined in §0.8.

Note that by definition, we have a morphism of complexes

$$R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet) \to R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, \mathcal{F}^\bullet). \tag{0.9.7}$$

*Indeed, let $v^*\mathcal{F}^\bullet \xrightarrow{\simeq} I^\bullet$ be a resolution of $v^*\mathcal{F}^\bullet$ by injective sheaves on $\operatorname{Spec}\mathbb{R}_{ét}$, and let $P_\bullet \twoheadrightarrow \mathbb{Z}$ be a resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}G$-modules. Then, thanks to the equivalence of categories $\mathbf{Sh}(\operatorname{Spec}\mathbb{R}_{ét})^{\mathbf{Ab}} \xrightarrow{\simeq} G_\mathbb{R}\text{-}\mathbf{Mod}$, the complex of $G_\mathbb{R}$-modules $I^\bullet_\mathbb{C}$ is an injective resolution of $(v^*\mathcal{F}^\bullet)_\mathbb{C} = \mathcal{F}_\mathbb{C}$. We have canonical quasi-isomorphisms of complexes

$$\operatorname{Hom}_{\mathbf{Sh}(\operatorname{Spec}\mathbb{R}_{ét})}(\underline{\mathbb{Z}}, I^\bullet) \to \operatorname{Tot}^\oplus \operatorname{Hom}_{\mathbb{Z}G}(P_\bullet, I^\bullet_\mathbb{C}) \leftarrow \operatorname{Tot}^\oplus \operatorname{Hom}^{\bullet\bullet}(P_\bullet, \mathcal{F}^\bullet_\mathbb{C}),$$

so in the derived category (!), there is an isomorphism

$$\operatorname{Hom}_{\mathbf{Sh}(\operatorname{Spec}\mathbb{R}_{ét})}(\underline{\mathbb{Z}}, I^\bullet) \xrightarrow{\cong} \operatorname{Tot}^\oplus \operatorname{Hom}^{\bullet\bullet}(P_\bullet, \mathcal{F}^\bullet_\mathbb{C}).$$

Now if $\mathcal{F}^\bullet$ is a bounded below complex of abelian sheaves on $X_{ét}$, then we pick a Nagata compactification of X

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathfrak{X} \arrow[dl, "g"] \\ & \operatorname{Spec}\mathbb{Z} & \end{tikzcd}$$

and set

$$R\widehat{\Gamma}_c(X_{ét}, \mathcal{F}^\bullet) := R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, Rf_!\mathcal{F}^\bullet),$$

where $Rf_! := Rg_* j_!$. In particular, the morphism (0.9.7) gives us for any bounded below complex of abelian sheaves $\mathcal{F}^\bullet$ on $X_{ét}$ a morphism

$$R\widehat{\Gamma}_c(X_{ét}, \mathcal{F}^\bullet) \to R\Gamma_c(X_{ét}, \mathcal{F}^\bullet), \tag{0.9.8}$$

where $R\Gamma_c(X_{ét}, \mathcal{F}^\bullet) := R\Gamma(\operatorname{Spec}\mathbb{Z}_{ét}, Rf_!\mathcal{F}^\bullet)$. By definition of $R\widehat{\Gamma}_c(\operatorname{Spec}\mathbb{Z}_{ét}, -)$, we have a long exact sequence in cohomology

$$\begin{aligned} \cdots \to \widehat{H}^{i-1}(G_\mathbb{R}, (Rf_!\mathcal{F}^\bullet)_\mathbb{C}) \to \widehat{H}^i_c(X_{ét}, \mathcal{F}^\bullet) \to H^i_c(X_{ét}, \mathcal{F}^\bullet) \\ \to \widehat{H}^i(G_\mathbb{R}, (Rf_!\mathcal{F}^\bullet)_\mathbb{C}) \to \cdots \end{aligned} \tag{0.9.9}$$

The groups $\widehat{H}^i(G_\mathbb{R}, (Rf_!\mathcal{F}^\bullet)_\mathbb{C})$ are annihilated by multiplication by $2 = \#G_\mathbb{R}$, which means that the morphism

$$\widehat{H}^i_c(X_{ét}, \mathcal{F}^\bullet) \to H^i_c(X_{ét}, \mathcal{F}^\bullet)$$

is identity, except for possible 2-torsion.

0.9.2. Remark. If $X(\mathbb{R}) = \emptyset$, then the canonical map

$$R\widehat{\Gamma}_c(X_{ét}, \mathcal{F}^*) \to R\Gamma_c(X_{ét}, \mathcal{F}^*)$$

is the identity.

0.10 Singular cohomology of complex varieties

We will need the following result.

0.10.1. Proposition. *Let X be an arithmetic scheme (separated, of finite type over* $\operatorname{Spec}\mathbb{Z}$*). Consider the corresponding space of complex points $X(\mathbb{C})$ equipped with the analytic topology. Then*

1) *the singular cohomology groups with compact support $H^i_c(X(\mathbb{C}), \mathbb{Z})$ are finitely generated for all i;*

2) *the groups* $H^i_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$ *are of cofinite type* ($\mathbb{Q}/\mathbb{Z}$*-dual of finitely generated groups).*

The above groups vanish for $i \gg 0$*.*

The statement is very plausible, but I could not find a good reference, so I outline a proof.

Proof. Everything relies on the fact that $X(\mathbb{C})$ has homotopy type of a finite CW-complex. This is a well-known classical result, due to van der Waerden (see [vdW1930] and [LW1933]).

If $X(\mathbb{C})$ is smooth, then we may reduce the problem to the case of pure dimension d, and by Poincaré duality,

$$H^i_c(X(\mathbb{C}), \mathbb{Z}) \cong H_{2d-i}(X(\mathbb{C}), \mathbb{Z}),$$

where $H_{2d-i}(X(\mathbb{C}), \mathbb{Z})$ are finitely generated groups, trivial for all but finitely many i, as $X(\mathbb{C})$ is homotopy equivalent to a finite CW-complex, and the homology $H_\bullet(X(\mathbb{C}), \mathbb{Z})$ is homotopy invariant.

To deal with the general case, we use induction on the dimension. If the dimension is 0, then the statement is obvious. For induction step, we may consider the open-closed decomposition

$$U(\mathbb{C}) \hookrightarrow X(\mathbb{C}) \hookleftarrow Z(\mathbb{C})$$

where $Z(\mathbb{C})$ is the singular locus, having smaller dimension. This gives us a distinguished triangle

$$R\Gamma_c(U(\mathbb{C}), \mathbb{Z}) \to R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \to R\Gamma_c(Z(\mathbb{C}), \mathbb{Z}) \to R\Gamma_c(U(\mathbb{C}), \mathbb{Z})[1]$$

where $R\Gamma_c(U(\mathbb{C}), \mathbb{Z})$ is a perfect complex by the above argument, and the complex $R\Gamma_c(Z(\mathbb{C}), \mathbb{Z})$ is perfect by induction. This implies that $R\Gamma_c(X(\mathbb{C}), \mathbb{Z})$ is a perfect complex.

As for $\mathbb{Q}/\mathbb{Z}$-coefficients, the statement follows from the distinguished triangle (keep in mind that tensoring with $\mathbb{Q}$ is exact)

$$R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \to R\Gamma_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to R\Gamma_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \to R\Gamma_c(X(\mathbb{C}), \mathbb{Z})[1]$$

Indeed, the associated long exact sequence in cohomology

$$\begin{aligned} \cdots \to H^i_c(X(\mathbb{C}), \mathbb{Z}) \to H^i_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^i_c(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}) \to \\ H^{i+1}_c(X(\mathbb{C}), \mathbb{Z}) \to H^{i+1}_c(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \cdots \end{aligned}$$

shows that $H^i_c(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z})$ is an extension of a finite group by a group of cofinite type, hence it is of cofinite type as well (see 0.1.3):

$$\begin{aligned} 0 \to \operatorname{coker}(H^i_c(X(\mathbb{C}),\mathbb{Z}) \to H^i_c(X(\mathbb{C}),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \to \\ H^i_c(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z}) \to \\ \ker(H^{i+1}_c(X(\mathbb{C}),\mathbb{Z}) \to H^{i+1}_c(X(\mathbb{C}),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \to 0 \end{aligned}$$

Finally, $H^i_c(X(\mathbb{C}),\mathbb{Q}/\mathbb{Z})$ vanishes for $i \gg 0$, because $H^i_c(X(\mathbb{C}),\mathbb{Z})$ does. ∎

0.11 Cycle complexes and motivic cohomology

Bloch's cycle complexes were introduced in [Blo1986a] to define higher Chow groups (there was a gap in the proof of the "moving lemma" that was fixed later in [Blo1994]). A good modern survey of cycle complexes may be found in [Gei2005], and there is also a useful text [Blo2005] available from Bloch's home page.

In this section I will go through various definitions that will be used later on in the constructions. Let X be an arithmetic scheme (separated, of finite type over $\operatorname{Spec}\mathbb{Z}$) or a variety over a field k (a separated scheme of finite type over $\operatorname{Spec} k$). Let $n \in \mathbb{Z}$ be some fixed integer. Then to X we may associate the following objects:

1a) a homological complex of abelian groups $z_n(X,\bullet)$, defined in terms of cycles of dimension $n+i$ in $X \times \Delta^i$;

1b) the corresponding cohomological complex of étale and Zariski sheaves

$$\mathbb{Z}^c(n) := z_n(-,-\bullet -2n);$$

2a) a homological complex of abelian groups $z^n(X,\bullet)$, defined in terms of cycles of codimension n in $X \times \Delta^i$, where Δ^i is the algebraic i-simplex;

2b) the corresponding cohomological complex of étale and Zariski sheaves

$$\mathbb{Z}(n) := z^n(-,2n-\bullet);$$

2c) some variation of 2a): a homological complex of abelian groups $z^n_{\square}(X,\bullet)$, defined in terms of cycles of codimension n in $X \times \square^i$, where $\square^i$ is the algebraic i-cube.

In fact, we will use only 1a) and 1b) in our constructions. I discuss 2a), 2b), 2c) simply because at some point (namely, in chapter 2) we will need to refer to the literature where 2a), 2b), 2c) are used instead of 1a) and 1b).

Simplicial and cubical complexes

Let us briefly recall some definitions regarding simplicial objects (see [May1992] and [GJ2009]) and cubical objects (see e.g. [Cis2006] and [BHS2011]). I do this mostly because of the cubical objects that seem to be less common.

0.11.1. Definition. The **simplicial category $\mathbf{\Delta}$** is the category where the objects are finite ordered sets

$$\mathbf{n} := \{0 < 1 < \cdots < n\}$$

for $n = 0, 1, 2, \ldots$ and the morphisms are nondecreasing maps $\mathbf{n} \to \mathbf{m}$.

A **simplicial** (resp. **cosimplicial**) **object** in a category $\mathcal{C}$ is a contravariant functor $X\colon \mathbf{\Delta}^\circ \to \mathcal{C}$ (resp. covariant functor $X\colon \mathbf{\Delta}^\circ \to \mathcal{C}$).

For $0 \le i \le n$, let us denote by

$$\partial_n^i\colon \mathbf{n-1} \hookrightarrow \mathbf{n}$$

the increasing map that skips i:

$$\partial_n^i(j) := \begin{cases} j, & j < i, \\ j+1, & j \ge i; \end{cases}$$

and let us denote by

$$\sigma_n^i\colon \mathbf{n+1} \twoheadrightarrow \mathbf{n}$$

the nondecreasing map that applies two elements to i:

$$\sigma_n^i(j) := \begin{cases} j, & j \le i, \\ j-1, & j > i. \end{cases}$$

Sometimes ∂^i's are called **coface morphisms** and σ^i's are called **codegeneracy morphisms**. It is easy to see that every morphism in $\mathbf{\Delta}$ may be written as a composition of such maps, and they satisfy the so-called **cosimplicial identitites**:

$$\sigma_n^j \circ \partial_{n+1}^i = \begin{cases} \partial_n^i \circ \sigma_{n-1}^{j-1}, & \text{if } i < j, \\ \mathrm{id}_{\mathbf{n}}, & \text{if } i = j \quad \text{or} \quad i = j+1, \\ \partial_n^{i-1} \circ \sigma_{n-1}^j, & \text{if } i > j+1; \end{cases} \tag{0.11.1}$$

$$\sigma_n^j \circ \sigma_{n+1}^i = \sigma_n^i \circ \sigma_{n+1}^{j+1} \quad \text{if } i \le j; \tag{0.11.2}$$

$$\partial_n^j \circ \partial_{n-1}^i = \partial_n^i \circ \partial_{n-1}^{j-1} \quad \text{if } i < j; \tag{0.11.3}$$

in fact, (0.11.1), (0.11.2), (0.11.3) give all possible relations between morphisms in $\mathbf{\Delta}$. This means that a simplicial object $X\colon \mathbf{\Delta}^\circ \to \mathcal{C}$ is equivalent to a collection of objects

$$X_n := X(\mathbf{n}) \in \operatorname{Ob}(\mathcal{C}) \quad (n = 0, 1, 2, \ldots)$$

and a collection of morphisms

$$\partial_i^n\colon X_n \to X_{n-1}, \quad \sigma_i^n\colon X_n \to X_{n+1} \quad (0 \le i \le n),$$

called **face** and **degeneracy morphisms** that satisfy the **simplicial identities** (dual to the identities (0.11.1), (0.11.2), (0.11.3)):

$$\partial_i^{n+1} \circ \sigma_j^n = \begin{cases} \sigma_{j-1}^{n-1} \circ \partial_i^n, & \text{if } i < j, \\ \mathrm{id}_{X_n}, & \text{if } i = j \quad \text{or} \quad i = j+1, \\ \sigma_j^{n-1} \circ \partial_{i-1}^n, & \text{if } i > j+1, \end{cases} \tag{0.11.4}$$

$$\sigma_i^{n+1} \circ \sigma_j^n = \sigma_{j+1}^{n+1} \circ \sigma_i^n \quad \text{if } i \le j. \tag{0.11.5}$$

$$\partial_i^{n-1} \circ \partial_j^n = \partial_{j-1}^{n-1} \circ \partial_i^n \quad \text{if } i < j. \tag{0.11.6}$$

A simplicial object may be visualized as a diagram

$$\begin{array}{ccccc} & & & \xleftarrow{\;\partial_0^2\;} & \\ & \xleftarrow{\;\partial_0^1\;} & & \xrightarrow{\;\sigma_0^2\;} & \\ X_0 & \xrightarrow{\;\sigma_0^1\;} & X_1 & \xleftarrow{\;\partial_1^2\;} & X_2 & \qquad \cdots \\ & \xleftarrow{\;\partial_1^1\;} & & \xrightarrow{\;\sigma_1^2\;} & \\ & & & \xleftarrow{\;\partial_2^2\;} & \end{array}$$

0.11.2. Lemma (Complex of alternating face maps). *Let $A\colon \mathbf{\Delta}^\circ \to \mathbf{Ab}$ be a simplicial abelian group. Then the morphisms of abelian groups*

$$d_n := \sum_{0 \le i \le n} (-1)^i \, \partial_i^n\colon A_n \to A_{n-1}$$

satisfy

$$d_{n-1} \circ d_n = 0,$$

i.e.

$$(A_\bullet, d_\bullet)\colon \quad \cdots \to A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \to 0$$

is a chain complex.

Proof. Easily follows from the simplicial identity (0.11.6). ∎

0.11.3. Definition ([Cis2006]). The **cubical category** $\square$ is the category where the objects are finite sets

$$\square^n := \{0,1\}^n = \{(x_1,\dots,x_n) \mid x_i \in \{0,1\}\}$$

for $n = 0,1,2,\dots$ and the morphisms are compositions of the following two kinds of maps:

1) for $n \geq 1$ and $1 \leq i \leq n$ the inclusion

$$\partial_n^{i,\epsilon}\colon \square^{n-1} \hookrightarrow \square^n$$

that inserts $\epsilon \in \{0,1\}$ into the i-th position:

$$\partial_n^{i,\epsilon}(x_1,\dots,x_{n-1}) := (x_1,\dots,x_{i-1},\epsilon,x_i,\dots,x_{n-1}). \tag{0.11.7}$$

2) for $n \geq 0$ and $1 \leq i \leq n+1$ the projection

$$\sigma_n^i\colon \square^{n+1} \twoheadrightarrow \square^n$$

that forgets the i-th coordinate:

$$\sigma_n^i(x_1,\dots,x_{n+1}) := (x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}). \tag{0.11.8}$$

A **cubical** (resp. **cocubical**) **object** in a category $\mathcal{C}$ is a contravariant functor $X\colon \square^\circ \to \mathcal{C}$ (resp. covariant functor $X\colon \square \to \mathcal{C}$).

All relations between the morphisms in $\square$ follow from the so-called **cocubical identities**:

$$\sigma_n^j \circ \partial_{n+1}^{i,\epsilon} = \begin{cases} \partial_n^{i,\epsilon} \circ \sigma_{n-1}^{j-1}, & \text{if } i < j, \\ \mathrm{id}_{\square^n}, & \text{if } i = j, \\ \partial_n^{i-1,\epsilon} \circ \sigma_{n-1}^{j}, & \text{if } i > j; \end{cases} \tag{0.11.9}$$

$$\sigma_n^j \circ \sigma_{n+1}^i = \sigma_n^i \circ \sigma_{n+1}^{j+1} \quad \text{if } i \leq j; \tag{0.11.10}$$

$$\partial_n^{j,\eta} \circ \partial_{n-1}^{i,\epsilon} = \partial_n^{i,\epsilon} \circ \partial_{n-1}^{j-1,\eta} \quad \text{if } i < j. \tag{0.11.11}$$

This means that a cubical object $X\colon \square^\circ \to \mathcal{C}$ is just a collection of objects

$$X_n := X(\square^n) \in \mathrm{Ob}(\mathbf{C}), \quad n = 0,1,2,\dots$$

and morphisms
$$\partial^n_{i,\epsilon}\colon X_n \to X_{n-1}, \quad \sigma^n_i\colon X_n \to X_{n+1}$$
that satisfy the **cubical identities**, i.e. the ones dual to (0.11.9), (0.11.10), (0.11.11):

$$\partial^{n+1}_{i,\epsilon} \circ \sigma^n_j = \begin{cases} \sigma^{n-1}_{j-1} \circ \partial^n_{i,\epsilon}, & \text{if } i < j, \\ \mathrm{id}_{X_n}, & \text{if } i = j, \\ \sigma^{n-1}_j \circ \partial^n_{i-1,\epsilon}, & \text{if } i > j; \end{cases} \tag{0.11.12}$$

$$\sigma^{n+1}_i \circ \sigma^n_j = \sigma^{n+1}_{j+1} \circ \sigma^n_i \quad \text{if } i \le j; \tag{0.11.13}$$

$$\partial^{n-1}_{i,\epsilon} \circ \partial^n_{j,\eta} = \partial^{n-1}_{j-1,\eta} \circ \partial^n_{i,\epsilon} \quad \text{if } i < j. \tag{0.11.14}$$

0.11.4. Lemma (Reduced cubical complex). *Let $A\colon \square^\circ \to \mathbf{Ab}$ be a cubical abelian group. Consider the morphisms*

$$d_n := \sum_{1\le i\le n} (-1)^i\,(\partial^n_{i,1} - \partial^n_{i,0})\colon A_n \to A_{n-1}. \tag{0.11.15}$$

Then

1) $d_{n-1} \circ d_n = 0$, *i.e.* $(A_\bullet, d_\bullet)$ *is a chain complex.*

2) *The **degenerate cubes** defined by*
$$(A_n)_{degn} := \sum_{1\le i\le n} \sigma^{n-1}_i(A_{n-1}) \subset A_n$$
form a subcomplex of $(A_\bullet, d_\bullet)$.

3) *We also have the subcomplex of **reduced cubes** given by*
$$(A_n)_0 := \bigcap_{1\le i\le n} \ker \partial^n_{i,1} \subset A_n.$$

4) *There is a canonical splitting*
$$A_n = (A_n)_{degn} \oplus (A_n)_0.$$

Sketch of the proof. Writing out all the involved combinatorial identities might not be very illuminating, but the reader should note how everything resembles the simplicial setting. 1) is deduced from the cubical identity (0.11.14);

in 2), to show that $d_n((A_n)_{degn}) \subseteq (A_{n-1})_{degn}$, one should use the cubical identity (0.11.12); in 3), to show that $d_n((A_n)_0) \subseteq (A_{n-1})_0$, one should again use (0.11.14). Finally, to show 4), one may consider the endomorphism $\pi_n \colon A_n \to A_n$ defined by

$$\pi_n := (\mathrm{id} - \sigma_n^{n-1} \circ \partial_{n,1}^n) \circ (\mathrm{id} - \sigma_{n-1}^{n-1} \circ \partial_{n-1,1}^n) \circ \cdots \\ \circ (\mathrm{id} - \sigma_2^{n-1} \circ \partial_{2,1}^n) \circ (\mathrm{id} - \sigma_1^{n-1} \circ \partial_{1,1}^n).$$

Then π_n defines the splitting

$$0 \to (A_n)_{degn} \to A_n \xrightarrow{\pi_n} (A_n)_0 \to 0$$

Namely, it is clear from the definition that $\pi_n|_{(A_n)_0} = \mathrm{id}_{(A_n)_0}$, and one deduces from the cubical identities that $\ker \pi_n = (A_n)_{degn}$ and $\operatorname{im} \pi_n = (A_n)_0$. ■

0.11.5. Definition. In the setting of 0.11.4, the **reduced cubical complex** associated to a cubical abelian group $A \colon ❒^\circ \to \mathbf{Ab}$ is the quotient complex

$$(A_\bullet/(A_\bullet)_{degn}, d_\bullet) \cong ((A_\bullet)_0, d_\bullet).$$

0.11.6. Remark (Cubical singular complex in topology). It is worth noting why quotienting out the degenerate cubes is necessary. Everything is motivated by cubical (co)homology in algebraic topology (see e.g. [Mas1977] and [EM1953]). We consider the **geometric cubes** defined for each $n = 0, 1, 2, 3, \ldots$ by

$$\square^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1\}.$$

We naturally have inclusions $\partial_n^{i,\epsilon} \colon \square^{n-1} \hookrightarrow \square^n$ and projections $\sigma_n^i \colon \square^{n-1} \twoheadrightarrow \square^n$, defined by the same formulas (0.11.7) and (0.11.8). This gives us a cocubical topological space $\square^\bullet \colon ❒ \to \mathbf{Top}$. Now for a topological space X, the sets

$$\mathrm{Sing}^\square(X)_n := \mathrm{Hom}_{\mathbf{Top}}(\square^n, X)$$

form a cubical set $\mathrm{Sing}^\square(X)_\bullet \colon ❒^\circ \to \mathbf{Set}$, which is the composition of functors $\square^\bullet \colon ❒ \to \mathbf{Top}$ and $\mathrm{Hom}_{\mathbf{Top}} \colon \mathbf{Top}^\circ \to \mathbf{Set}$. Namely, for a continuous map $\phi \colon \square^n \to X$, we may consider its restrictions to $\square^{n-1} \subset \square^n$ given by setting $x_i = 0$ or $x_i = 1$ for $i = 1, \ldots, n$, and also extensions to $\square^{n+1} \supset \square^n$ given by putting 0 or 1 in i-th position. This gives us face and degeneracy maps

$$\partial_{i,\epsilon} \colon \mathrm{Sing}^\square(X)_n \to \mathrm{Sing}^\square(X)_{n-1},$$
$$\sigma_i \colon \mathrm{Sing}^\square(X)_n \to \mathrm{Sing}^\square(X)_{n+1}$$

that satisfy the cubical identities. By composing our functor $\mathrm{Sing}^\square(X)_\bullet \colon ❒^\circ \to \mathbf{Set}$ with the free abelian group functor $\mathbf{Set} \to \mathbf{Ab}$, we obtain a cubical abelian group $\mathbb{Z}\left\langle \operatorname{Sing}^{\square}(X)_{\bullet}\right\rangle : \boldsymbol{\square}^{\circ} \to \mathbf{Ab}$. As in 0.11.4, we may build from it a chain complex.

Now if $X = *$ is just a point, then

$$\operatorname{Sing}^{\square}(*)_n = \operatorname{Hom}_{\mathbf{Top}}(\square^n, *)$$

are one-element sets, so that the complex will look like

$$\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

However, note that in this case we have $\partial^n_{i,1} = \partial^n_{i,0}$ for all n and i, therefore the differentials (0.11.15) are all trivial, and the point has homology $\cong \mathbb{Z}$ in all degrees, which is not very desirable. However, the cubes of dimension $n > 0$ are all degenerate, so the corresponding *reduced* cubical complex looks like

$$\cdots \to 0 \to 0 \to 0 \to \mathbb{Z} \to 0$$

Note that for the usual singular complex defined using simplices instead of cubes (replace $\square^n$ with Δ^n in all the above), the degenerate simplices also form a subcomplex, but it is easily seen from the simplicial identities that passing to the corresponding reduced complex does not affect the homology. E.g. the simplicial singular complex for a point will look like

$$\cdots \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

For comparison of the simplicial and cubical approach to defining singular (co)homology, see [EM1953].

Bloch's cycle complexes $z^n(X, \bullet)$ and $z_n(X, \bullet)$

Now we define several versions of Bloch's cycle complexes; we refer to [Gei2005] and [Blo2005] for details; our reference for intersection theory is [Ful1998].

Let X be a separated scheme of finite type over a base scheme S. For our particular purposes, we only consider the cases $S = \operatorname{Spec} k$ for a field k or $S = \operatorname{Spec} \mathbb{Z}$. For each $n = 0, 1, 2, \ldots$ the **algebraic n-simplex** is given by

$$\Delta^n := \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_n]/(1 - \sum_i t_i), \quad \Delta^n_S := \Delta^n \times_{\operatorname{Spec} \mathbb{Z}} S.$$

This is isomorphic to the affine space $\mathbb{A}^n$, but not canonically; instead, it comes with canonical "simplicial" coordinates. Each nondecreasing map

$\rho\colon \mathbf{n} \to \mathbf{m}$ induces functorially a morphism of schemes $\widetilde{\rho}\colon \Delta^n_S \to \Delta^m_S$ given by

$$\widetilde{\rho}(t_i) := \begin{cases} 0, & \text{if } \rho^{-1}(i) = \emptyset, \\ \sum_{\rho(j)=i} t_j, & \text{otherwise.} \end{cases}$$

Similarly, for an S-scheme X, a nondecreasing map $\rho\colon \mathbf{n} \to \mathbf{m}$ induces functorially a morphism of schemes $(\mathrm{id} \times \widetilde{\rho})\colon X \times_S \Delta^n_S \to X \times_S \Delta^m_S$. This defines a cosimplicial S-scheme

$$X \times_S \Delta^\bullet_S\colon \mathbf{\Delta} \to \mathbf{Sch}_{/S}.$$

Now for a fixed $n \in \mathbb{Z}$, one considers the following two series of free abelian groups indexed by $i \in \mathbb{Z}$:

1) we let $z_n(X, i)$ be the free abelian group generated by closed integral subschemes $Z \subset X \times_S \Delta^i_S$ of dimension $n + i$ that meet all faces of Δ^i_S properly.

2) if X is an equidimensional scheme, we let $z^n(X, i)$ be the free abelian group generated by closed integral subschemes $Z \subset X \times \Delta^i$ of *codimension n* that meet all faces of Δ^i_S properly.

We note that the first definition is in fact more natural in some sense: it does not require X to be equidimensional. If X is of pure dimension d, then we see that

(0.11.16) $$z_n(X, i) = z^{d-n}(X, i).$$

Both $z_n(X, \bullet)$ and $z^n(X, \bullet)$ are simplicial abelian groups $\mathbf{\Delta}^\circ \to \mathbf{Ab}$. Namely, for a morphism $\rho\colon \mathbf{i} \to \mathbf{j}$ in $\mathbf{\Delta}$,

1) if ρ is injective, then $\mathrm{id} \times \widetilde{\rho}\colon X \times_S \Delta^i_S \to X \times_S \Delta^j_S$ is a closed immersion, and for a cycle $V \subset X \times_S \Delta^j_S$ we may consider the intersection
$$(\mathrm{id} \times \widetilde{\rho})(X \times_S \Delta^i_S) \cdot V;$$

2) if ρ is surjective, then $\mathrm{id} \times \widetilde{\rho}\colon X \times_S \Delta^i_S \to X \times_S \Delta^j_S$ is a flat morphism, for which we have the corresponding flat pullback of cycles;

in both cases, we obtain morphisms

$$\rho^*\colon z_n(X, j) \to z_n(X, i), \quad \rho^*\colon z^n(X, j) \to z^n(X, i).$$

In particular, as we noted in 0.11.2, $z_n(X, \bullet)$ and $z^n(X, \bullet)$ give us chain complexes

$$\cdots \to z_n(X, i) \xrightarrow{d_i} z_n(X, i-1) \xrightarrow{d_{i-1}} z_n(X, i-2) \to \cdots$$

and

$$\cdots \to z^n(X,i) \xrightarrow{d_i} z^n(X,i-1) \xrightarrow{d_{i-1}} z^n(X,i-2) \to \cdots$$

with the differentials

$$d_i := \sum_{0\le \ell \le i} (-1)^\ell\, \partial_\ell.$$

Let us recall Bloch's definition of higher Chow groups, for which he introduced the complexes $z^n(X,\bullet)$.

0.11.7. Definition ([Blo1986a]). If X is an equidimensional scheme as above, its **higher Chow groups** are given by

$$CH^n(X,i) := H_i(z^n(X,\bullet)).$$

The usual Chow groups (algebraic cycles on X modulo rational equivalence) correspond to $i = 0$:

$$CH^n(X) = CH^n(X,0).$$

Cubical cycle complexes $z^n_\square(X,\bullet)$

We briefly recall the cubical version of $z^n(X,\bullet)$, which is often used in the literature, e.g. in [Lev1994]. If k is a field, we consider the **algebraic cube**

$$\square^n_k := (\mathbb{P}^1_k \setminus \{1\})^n$$

with coordinates $(x_1,\ldots,x_n)$. Setting some x_i to 0 or ∞ gives us a **codimension 1 face of** $\square^n_k$. In general, setting $x_{i_1},\ldots,x_{i_s}$ to 0 or ∞ gives a **codimension s face**. We have a cocubical variety $\square^\bullet_k$ in the sense of 0.11.3. Namely,

1) for each $n \ge 1$ we have the inclusion maps

$$\begin{gathered}\partial^{i,\epsilon}_n\colon \square^{n-1}_k \hookrightarrow \square^n_k, \quad (1\le i\le n,\ \epsilon = 0,\infty)\\ (x_1,\ldots,x_{n-1}) \mapsto (x_1,\ldots,x_{i-1},\epsilon,x_i,\ldots,x_{n-1});\end{gathered}$$

2) for $n \ge 0$ we have the projection maps

$$\begin{gathered}\sigma^i_n\colon \square^{n+1}_k \twoheadrightarrow \square^n_k \quad (1\le i\le n+1),\\ (x_1,\ldots,x_{n+1}) \mapsto (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1});\end{gathered}$$

and these maps satisfy the cocubical identities.

Now if X is an equidimensional variety over k, we denote by $z^n_\square(X,i)$ the free abelian group generated by the irredicible subvarieties

$$V \subset X \times_k \square^i_k, \quad \operatorname{codim}_k V = n,$$

meeting all faces properly. The maps $\mathrm{id} \times \partial_i^{\ell,\epsilon}$ and $\mathrm{id} \times \sigma_i^\ell$ induce pullback morphisms

$$\partial^i_{\ell,\epsilon} \colon z^n_\square(X,i) \to z^n_\square(X,i-1),$$
$$\sigma^i_\ell \colon z^n_\square(X,i) \to z^n_\square(X,i+1)$$

which satisfy the cubical identities and thus turn $z^n_\square(X,\bullet)$ into a cubical abelian group. As in 0.11.5, we form from this a chain complex $z^n_\square(X,\bullet)$, where the differentials are given by

$$d_i := \sum_{1\le\ell\le i} (-1)^\ell\,(\partial^i_{\ell,\infty} - \partial^i_{\ell,0}) \colon z^n_\square(X,i) \to z^n_\square(X,i-1),$$

and the degenerate cubes are quotiented out. The following is proved in [Lev1994].

0.11.8. Theorem. *There is an isomorphism in the derived category*

$$z^r_\square(X,\bullet) \cong z^r(X,\bullet).$$

Complexes of sheaves $\mathbb{Z}^c(n)$

The cycle complexes may be "sheafified" as follows. The presheaves

$$U \mapsto z_n(U,i), \quad U \mapsto z^n(U,i)$$

are in fact sheaves on $X_{ét}$ or X_{Zar} (this is verified e.g. in [Gei2004, Lemma 3.1]). We will use the opposite numbering and denote

$$\mathcal{Z}_n^X := z_n(-,-\bullet), \quad \mathcal{Z}_X^n := z^n(-,-\bullet).$$

These are cohomological complexes of abelian sheaves on $X_{ét}$ or X_{Zar}.

We will also need the following result, saying that the cohomology of the cycle complexes $z^n(X,-\bullet)$ coincides with the Zariski hypercohomology of $\mathcal{Z}_X^n$.

0.11.9. Theorem. *If X is a scheme of finite type over a field, we have a quasi-isomorphism of complexes of abelian groups*

$$R\Gamma(X_{Zar},\mathcal{Z}_X^n) \simeq z^n(X,-\bullet).$$

Proof. See [Gei2005, §1.2.4.] for details. ∎

Finally, in terms of $\mathcal{Z}_n^X$, one defines complexes $\mathbb{Z}^c(n)$, which will be one of the most important objects in our constructions.

0.11.10. Definition ([Gei2010]). The **dualizing cycle complex** is given by

$$\mathbb{Z}^c(n) := \mathcal{Z}_n^X[2n].$$

It is a cohomological complex of sheaves with $z_n(-, -i-2n)$ sitting in i-th degree. In general, for any abelian group A, one defines

$$A^c(n) := \mathbb{Z}^c(n) \otimes^{\mathbf{L}}_{\mathbb{Z}} A = \mathbb{Z}^c(n) \otimes_{\mathbb{Z}} A.$$

(As $\mathbb{Z}^c(n)$ is a complex of flat sheaves, the derived tensor product coincides with the usual tensor product.)

For the sake of completeness, I also recall the the related definition based on $z^n(-, \bullet)$:

$$\mathbb{Z}(n) := \mathcal{Z}^n_X[-2n].$$

It is a cohomological complex of sheaves with $z^n(-, 2n-i)$ in degree i. If X is equidimensional of dimension d, then (0.11.16) gives us the corresponding relation for complexes of sheaves

$$\mathcal{Z}_n^X = \mathcal{Z}_X^{d-n},$$

which allows us to express $\mathbb{Z}^c(n)$ in terms of $\mathcal{Z}^n_X$:

$$\mathbb{Z}^c(n) = \mathcal{Z}_X^{d-n}[2n] = \mathbb{Z}(d-n)[2d].$$

Now the reader should actually forget about this $\mathbb{Z}(n)$, because later on "$\mathbb{Z}(n)$" will denote a completely different complex of sheaves, to be defined in §1.2.

$\mathbb{Z}^c(n)$ as a dualizing complex

0.11.11. Topological digression. Let us recall that for a locally compact topological space X, one may define **Borel–Moore homology groups** $H_i^{BM}(X, \mathbb{Z})$ (see [Ive1986, Chapter IX]). These will make their appearance in §2.1, but now they will serve us as a motivating example of duality.

Local Verdier duality [Ive1986, §VII.5] tells that if $f\colon X \to Y$ is a continuous map between locally compact topological spaces of finite dimension, then there is a natural isomorphism in the derived category $\mathbf{D}^+(Y)$

$$R\underline{\mathrm{Hom}}(Rf_! \mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong Rf_* R\underline{\mathrm{Hom}}(\mathcal{F}^\bullet, f^! \mathcal{G}^\bullet)$$

where $\mathcal{F}^\bullet \in \mathbf{D}^+(X)$, $\mathcal{G}^\bullet \in \mathbf{D}^+(Y)$, and $f^!\colon \mathbf{D}^+(Y) \to \mathbf{D}^+(X)$ is the right adjoint functor to $Rf_!\colon \mathbf{D}^+(X) \to \mathbf{D}^+(Y)$. In particular, for the projection to the point $p\colon X \to *$ the above reads

$$R\mathrm{Hom}(R\Gamma_c(X, \mathcal{F}^\bullet), \mathcal{G}^\bullet) \cong R\Gamma(X, R\underline{\mathrm{Hom}}(\mathcal{F}^\bullet, p^! \mathcal{G}^\bullet))$$

for $\mathcal{F} \in \mathbf{D}^+(X)$ and $\mathcal{G}^\bullet \in \mathbf{D}(\mathbf{Ab})$. If we take $\mathcal{G}^\bullet$ to be the complex consisting of a single constant sheaf $\mathbb{Z}$, the object $p^!\mathbb{Z} \in \mathbf{D}^+(X)$ is called the **dualizing sheaf** on X, and **Borel–Moore homology** is defined by

$$\begin{aligned} H_i^{BM}(X,\mathbb{Z}) &:= H^{-i}(R\Gamma_{BM}(X,\mathbb{Z})), \\ R\Gamma_{BM}(X,\mathbb{Z}) &:= R\Gamma(X, p^!\mathbb{Z}) \cong R\mathrm{Hom}(R\Gamma_c(X,\mathbb{Z}),\mathbb{Z}). \end{aligned}$$

This means that Borel–Moore homology is *covariantly* functorial for proper maps and *contravariantly* functorial for inclusions of open subsets $U \hookrightarrow X$:

1) a proper continuous map of locally compact topological spaces $f\colon X \to Y$ induces a morphism $R\Gamma_c(Y,\mathbb{Z}) \to R\Gamma_c(X,\mathbb{Z})$, and therefore on Borel–Moore homology we have the **proper pushforward** morphism

$$R\Gamma_{BM}(X,\mathbb{Z}) \to R\Gamma_{BM}(Y,\mathbb{Z}).$$

2) an inclusion of an open subset $U \hookrightarrow X$ induces a morphism $R\Gamma_c(U,\mathbb{Z}) \to R\Gamma_c(X,\mathbb{Z})$, and therefore the corresponding **pullback** on Borel–Moore homology

$$R\Gamma_{BM}(X,\mathbb{Z}) \to R\Gamma_{BM}(U,\mathbb{Z}).$$

Moreover, if $U \subset X$ is an open subset and $Z := X \setminus U$ is its closed complement, then the corresponding pushforwards and pullbacks fit into a distinguished triangle

$$R\Gamma_{BM}(Z,\mathbb{Z}) \to R\Gamma_{BM}(X,\mathbb{Z}) \to R\Gamma_{BM}(U,\mathbb{Z}) \to R\Gamma_{BM}(X,\mathbb{Z})[1]$$

This is dual to the triangle

$$R\Gamma_c(U,\mathbb{Z}) \to R\Gamma_c(X,\mathbb{Z}) \to R\Gamma_c(Z,\mathbb{Z}) \to R\Gamma_c(U,\mathbb{Z})[1]$$

The cycle complex $\mathbb{Z}^c(n)$ behaves similarly to Borel–Moore homology.

0.11.1. Fact ([Gei2010, Corollary 7.2]).

1) a proper morphism of schemes $f\colon X \to Y$ induces a pushforward morphism

$$Rf_*\mathbb{Z}^c_X(n) \to \mathbb{Z}^c_Y(n);$$

2) an open immersion of schemes $f\colon U \hookrightarrow X$ induces a flat pullback morphism

$$f^*\mathbb{Z}^c_X(n) \to \mathbb{Z}^c_U(n).$$

If $U \subset X$ is an open subscheme and $Z := X \setminus U$ is its closed complement, then the proper pushforward associated to $Z \hookrightarrow X$ and the flat pullback associated to $U \hookrightarrow X$ give a distinguished triangle

$$R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)) \to R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n))[1]$$

Chapter 1

Weil-étale complexes

For an arithmetic scheme X (separated, of finite type over $\operatorname{Spec}\mathbb{Z}$) and a strictly negative integer n, we are going to construct certain complexes $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, following Flach and Morin [Mor2014, FM2016]. Here "W" stays for "Weil-étale" and "c" stays for "compact support".

The constructions are based on complexes of sheaves $\mathbb{Z}^c(n)$ on $X_{ét}$, discussed in §0.11. The basic properties of motivic cohomology for arithmetic schemes are still conjectural, and in order to make sense of all our constructions, we will need to assume in 1.1.1 that the groups $H^i(X_{ét},\mathbb{Z}^c(n))$ are finitely generated.

It is worth mentioning that the constructions in [FM2016] use other cycle complexes $\mathbb{Z}(n)$, mentioned in §0.11. If X has pure dimension d, then all this amounts to the renumbering

$$\mathbb{Z}^c(n) = \mathbb{Z}(d-n)[2d], \tag{1.0.1}$$

which should be taken into account when comparing formulas that will appear below with the formulas from [FM2016]. We use $\mathbb{Z}^c(n)$ instead of $\mathbb{Z}(n)$ precisely to avoid any references to the dimension of X (which is not assumed anymore to be equidimensional). Indeed, the dimensions of cohomology groups in many formulas in [FM2016] have terms "$2d$", and if one rewrites everything using (1.0.1), they magically disappear. This suggests that $\mathbb{Z}^c(n)$ is a more natural object than $\mathbb{Z}(n)$ in our situation.

In fact, §1.2 introduces a special definition of $\mathbb{Z}(n)$, motivated by [FM2016], which is unrelated to the cycle complexes. In our setting $n < 0$, the complex $\mathbb{Z}(n)$ will consist of a single étale sheaf, rather easy to define and understand.

Both $\mathbb{Z}^c(n)$ and $\mathbb{Z}(n)$ will appear in a certain arithmetic duality theorem in §1.3, which is stated as a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

In §1.4 I take a look at $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))$ and related complexes. Then using the duality theorem, I define in §1.5 a morphism in the derived category $\mathbf{D}(\mathbf{Ab})$

$$\alpha_{X,n}\colon R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\Gamma_c(X_{ét}, \mathbb{Z}(n))$$

and declare $R\Gamma_{fg}(X, \mathbb{Z}(n))$ to be its cone:

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ \to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{aligned}$$

The complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is **almost perfect** in the sense of 0.3.3 (i.e. a perfect complex modulo possible 2-torsion in arbitrarily high degrees), canonical and functorial (despite being defined as a cone in the derived category).

Then §1.6 is dedicated to the definition of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. For this we will need a morphism

$$i_\infty^*\colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}),$$

where $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ stays for the $G_\mathbb{R}$-equivariant cohomology with compact support on $X(\mathbb{C})$. Then $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ will be given (sadly, up to a non-unique isomorphism in $\mathbf{D}(\mathbf{Ab})$) by the distinguished triangle

$$\begin{aligned} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \\ \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \end{aligned}$$

The sheaf $(2\pi i)^n\,\mathbb{Z}$ is the constant $G_\mathbb{R}$-equivariant sheaf on $X(\mathbb{C})$, which is the image of $\mathbb{Z}(n)$ under the morphism α^* from §0.7 (see 1.6.2). The existence of i_∞^* relies on a rather nontrivial argument (theorem 1.6.4).

I show in §1.7 that there is a (non-canonical) splitting

$$\begin{aligned} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_\mathbb{Z} \mathbb{Q} \cong \\ R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1]. \end{aligned}$$

Finally, §1.8 is dedicated to verifying that $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is well-behaved with respect to open-closed decompositions of schemes $U \hookrightarrow X \leftarrow Z$. With the present definition, this cannot be shown for the complex itself, but we are going to establish a canonical isomorphism of the determinants

$$\det_\mathbb{Z} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong \det_\mathbb{Z} R\Gamma_{W,c}(U, \mathbb{Z}(n)) \otimes_\mathbb{Z} \det_\mathbb{Z} R\Gamma_{W,c}(Z, \mathbb{Z}(n)),$$

which will be enough for our purposes.

1.1 Conjecture $\mathbf{L}^c(X_{ét}, n)$

Practically all our constructions will make use of the following hypothesis for an arithmetic scheme X and a strictly negative integer $n < 0$.

1.1.1. Conjecture $\mathbf{L}^c(X_{ét}, n)$. The groups $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.

This is analogous to "$\mathbf{L}(X_{ét}, n)$" (Conjecture 3.2) in [FM2016], but in our setting we need a statement for the dualizing cycle complexes $\mathbb{Z}^c(n)$. As we are going to see in 1.5.3, the conjecture $\mathbf{L}^c(X_{ét}, n)$ actually implies that for any arithmetic scheme X the complex $\mathbb{Z}^c(n)$ is bounded from below and has some finite 2-torsion in higher degrees. This is related to the **Beilinson–Soulé vanishing conjecture**, which has not been proved yet.

1.2 Complexes of étale sheaves $\mathbb{Z}(n)$ for $n < 0$

For our construction, we need to make sense of "cycle complexes" $\mathbb{Z}(n)$ for $n < 0$. Here we recall a good definition of such an object, coming from [FM2016, §6.2].

First of all, if $\mathbb{Z}(n)$ is defined, then for any abelian group A and $n \geq 0$, one can define the corresponding complex with coefficients in A by

$$A(n) := \mathbb{Z}(n) \otimes^{\mathbf{L}}_{\mathbb{Z}} A.$$

The usual distinguished triangle

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}[1]$$

should give after tensoring with $\mathbb{Z}(n)$ a distinguished triangle of complexes of sheaves

$$\mathbb{Q}/\mathbb{Z}(n)[-1] \to \mathbb{Z}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}(n)$$

and we can use this to define the cycle complex $\mathbb{Z}(n)$ for $n < 0$. In this case we should have $\mathbb{Q}(n) = 0$, so the triangle above suggests that we should put

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] \quad \text{for } n < 0.$$

The complex $\mathbb{Q}/\mathbb{Z}(n)$ still does not make sense for $n < 0$, but we should have something like

$$\mathbb{Q}/\mathbb{Z}(n) = \bigoplus_p \mathbb{Z}/p^\infty\mathbb{Z}(n) = \bigoplus_p \varinjlim_r \mathbb{Z}/p^r\mathbb{Z}(n),$$

and we define for $n < 0$

$$\mathbb{Z}/p^r\mathbb{Z}(n) := j_{p!}\mu_{p^r}^{\otimes n},$$

where

1) j_p is the open immersion $X[1/p] \to X$, and $j_{p!}\colon \mathbf{Sh}(X[1/p]_{ét}) \to \mathbf{Sh}(X_{ét})$ denotes the extension by zero functor;

2) μ_{p^r} is the sheaf of roots of unity on $X[1/p]_{ét}$ represented by the commutative group scheme

$$X[1/p] \underset{\operatorname{Spec}\mathbb{Z}[1/p]}{\times} \operatorname{Spec}\mathbb{Z}[1/p][t]/(t^{p^r}-1) \to X[1/p];$$

3) $\mu_{p^r}^{\otimes n}$ is the sheaf on $X[1/p]_{ét}$ defined by

$$\mu_{p^r}^{\otimes n} := \underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r).$$

Therefore we are going to use the following definition.

1.2.1. Definition. For each $n < 0$ we consider the complex of sheaves on $X_{ét}$

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1].$$

1.3 An Artin–Verdier-like duality

At the heart of our constructions is a certain arithmetic duality theorem for cycle complexes obtained by Thomas Geisser in [Gei2010]. It generalizes the classical Artin–Verdier duality (originating from one of the Woods Hole seminars [AV1964]; one of the few thorough discussions in the literature is the second chapter of Milne's book [Mil2006]).

1.3.1. Proposition ("Artin–Verdier duality"). *For any $n < 0$ we have a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \cong \varinjlim_m R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Proof. We unwind our definition of $\mathbb{Z}(n)$ for $n < 0$ and reduce everything to the results from [Gei2010]. It is worth remarking that Geisser uses notation "$R\Gamma_c$" for our "$R\widehat{\Gamma}_c$" (see §0.9).

As we have $\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1]$, it will be enough to show that for every prime p and $r = 1, 2, 3, \ldots$ there is a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]) \cong R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)), \mathbb{Q}/\mathbb{Z}[-2]),$$

and then pass to the corresponding filtered colimits.

As in §1.2, the morphism $j_p \colon X[1/p] \hookrightarrow X$ denotes the canonical open immersion. We further denote by $f \colon X \to \operatorname{Spec} \mathbb{Z}$ the structure morphism of X and by f_p the morphism $X[1/p] \to \operatorname{Spec} \mathbb{Z}[1/p]$:

$$\begin{tikzcd} X[1/p] \arrow[r, hook, "j_p"] \arrow[d, "f_p"'] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{Z}[1/p] \arrow[r, hook] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

As we are going to change the base scheme, let us write "$\operatorname{Hom}_X(-,-)$" for the Hom between sheaves on $X_{ét}$ (and "$\underline{\operatorname{Hom}}_X(-,-)$" for the internal Hom). Instead of "$\operatorname{Hom}_{\operatorname{Spec} R}$", we will simply write "Hom_R".

By [Gei2010, Proposition 7.10, (c)], we have the following "exchange formulas". If we work with complexes of constructible sheaves on the étale site of schemes over the spectrum of a number ring $\operatorname{Spec} O_K$, then for a morphism ϕ of such schemes we have

$$R\phi_* \mathcal{D}(\mathcal{F}) \cong \mathcal{D}(R\phi_! \mathcal{F}), \tag{1.3.1}$$

$$R\phi^! \mathcal{D}(\mathcal{G}) \cong \mathcal{D}(\phi^* \mathcal{G}), \tag{1.3.2}$$

where the dualization is given by

$$\mathcal{D}(\mathcal{F}^\bullet) := R\underline{\operatorname{Hom}}_X(\mathcal{F}^\bullet, \mathbb{Z}^c(0)).$$

Applying the exchange formula (1.3.1) to our situation, we get

$$R\underline{\operatorname{Hom}}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong Rj_{p*}R\underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)). \tag{1.3.3}$$

Using the other exchange formula (1.3.2), we may identify the sheaf $R\underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0))$:

$$\begin{aligned} R\underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)) &\cong R\underline{\operatorname{Hom}}_{X[1/p]}(f_p^*\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{X[1/p]}(0)) && (1.3.4)\\ &\cong Rf_p^! R\underline{\operatorname{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_{\mathbb{Z}[1/p]}(0)) && (1.3.5)\\ &\cong Rf_p^! R\underline{\operatorname{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}[-1], \mathbb{G}_m[1]) && (1.3.6)\\ &\cong Rf_p^! R\underline{\operatorname{Hom}}_{\mathbb{Z}[1/p]}(\mu_{p^r}^{\otimes n}, \mathbb{G}_m)[2] && (1.3.7)\\ &\cong Rf_p^! \mu_{p^r}^{\otimes(1-n)}[2] && (1.3.8) \end{aligned}$$

Here (1.3.4) simply means that the sheaf $\mu_{p^r}^{\otimes n}$ on $X[1/p]$ is the same as the inverse image of the corresponding sheaf on $\operatorname{Spec} \mathbb{Z}[1/p]$. The quasi-isomorphism (1.3.5) is the first exchange formula. Then, (1.3.6) is the fact that the complex $\mathbb{Z}^c_{\mathbb{Z}[1/p]}(0)$ is quasi-isomorphic to $\mathbb{G}_m[1]$ according to [Gei2010, Lemma 7.4]. Thanks to [Gei2004, Theorem 1.2], we may identify the sheaf $\mu_{p^r}^{\otimes(1-n)}$:

$$\mu_{p^r}^{\otimes(1-n)} \cong \mathbb{Z}_{\mathbb{Z}[1/p]}/p^r(1-n) = \mathbb{Z}^c_{\mathbb{Z}[1/p]}/p^r(n)[-2]. \tag{1.3.9}$$

Then [Gei2010, Corollary 7.9] tells us that

$$Rf_p^! \mathbb{Z}^c_{\mathbb{Z}[1/p]}/p^r(n) \cong \mathbb{Z}^c_{X[1/p]}/p^r(n). \tag{1.3.10}$$

Finally, thanks to [Gei2010, Theorem 7.2 (a)] and [Gei2010, Proposition 2.3], we have $\mathbb{Z}^c_{X[1/p]}/p^r(n) \cong j_p^* \mathbb{Z}^c_X/p^r(n)$, and all the above gives

$$R\underline{\mathrm{Hom}}_X(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong Rj_{p*}j_p^*\mathbb{Z}^c_X/p^r(n) \cong \mathbb{Z}^c_X/p^r(n). \tag{1.3.11}$$

After applying $R\Gamma(X_{ét}, -)$, we get a quasi-isomorphism of complexes of abelian groups

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c_X(0)) \cong R\Gamma(X_{ét}, \mathbb{Z}^c_X/p^r(n)). \tag{1.3.12}$$

Now according to the generalization of Artin–Verdier duality by Geisser [Gei2010, Theorem 7.8], we have

(1.3.13)

$$R\mathrm{Hom}(j_{p!}\mu_{p^r}^{\otimes n}[-1], \mathbb{Z}^c(0)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

So what we obtain at the end is a quasi-isomorphism

$$R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)) \cong R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]).$$

This is almost what we need: if we apply $R\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z}[-2])$, then, as $\widehat{H}^i_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1])$ are finite groups (because the sheaves $j_{p!}\mu_{p^r}^{\otimes n}$ are constructible), we have

$$\begin{aligned} &R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c/p^r(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong \\ &\qquad R\mathrm{Hom}(R\mathrm{Hom}(R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]), \mathbb{Q}/\mathbb{Z}[-2]), \mathbb{Q}/\mathbb{Z}[-2]) \\ &\qquad\qquad \cong R\widehat{\Gamma}_c(X_{ét}, j_{p!}\mu_{p^r}^{\otimes n}[-1]). \end{aligned}$$

■

The quasi-isomorphism

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \cong \varinjlim_m R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2])$$

that we just saw means that on the level of cohomology, we get

$$\widehat{H}^i_c(X_{ét}, \mathbb{Z}(n)) \cong \varinjlim_m \operatorname{Hom}(H^{2-i}(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z})$$

(note that the group $\mathbb{Q}/\mathbb{Z}$ is divisible, so $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor, and the filtered colimit $\varinjlim_m$ is exact as well).

1.3.2. Proposition. *Assuming the conjecture $\mathbf{L}^c(X_{ét}, n)$ (see 1.1.1), there is a quasi-isomorphism of complexes*

$$\varinjlim_m R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \cong \\ R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

Proof. As $\mathbb{Z}^c(n)$ is a complex of flat sheaves, the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

induces a short exact sequence of sheaves

$$0 \to \mathbb{Z}^c(n) \xrightarrow{\times m} \mathbb{Z}^c(n) \to \mathbb{Z}/m\mathbb{Z}^c(n) \to 0 \tag{1.3.14}$$

The morphism $\mathbb{Z}^c(n) \to \mathbb{Z}/m\mathbb{Z}^c(n)$ induces some morphisms in cohomology

$$H^i(X_{ét}, \mathbb{Z}^c(n)) \to H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)).$$

We claim that if we pass to the duals $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and then to the filtered colimits $\varinjlim_m$, then we obtain an isomorphism. (Note that both $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and $\varinjlim_m$ are exact.)

The short exact sequence (1.3.14) induces a long exact sequence in cohomology

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^i(X_{ét}, \mathbb{Z}^c(n)) \arrow[r, "\times m"] & H^i(X_{ét}, \mathbb{Z}^c(n)) \arrow[r] & H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \arrow[dll, "\delta^i"] \\
& H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \arrow[r, "\times m"] & H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \arrow[r] & H^{i+1}(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \arrow[r] & \cdots
\end{tikzcd}$$

We further have exact sequences

$$\begin{tikzcd}
& & \ker \delta^i \arrow[d, equal] & \\
H^i(X_{ét}, \mathbb{Z}^c(n)) \arrow[r, "\times m"] & H^i(X_{ét}, \mathbb{Z}^c(n)) \arrow[r] & H^i(X_{ét}, \mathbb{Z}^c(n))_m \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & {}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \arrow[r] \arrow[d, equal] & H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \arrow[r, "\times m"] & H^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \\
& \operatorname{im} \delta^i & &
\end{tikzcd}$$

that give us

$$0 \to H^i(X_{ét}, \mathbb{Z}^c(n))_m \to H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)) \to {}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)) \to 0$$

Now if we take $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ and filtered colimits $\varinjlim_m$, we get

$$(1.3.15) \quad 0 \to \varinjlim_m \operatorname{Hom}({}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to$$
$$\varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \to \varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \to 0$$

By the conjecture $\mathbf{L}^c(X_{ét}, n)$, the group $H^{i+1}(X_{ét}, \mathbb{Z}^c(n))$ is finitely generated, and therefore

$${}_mH^{i+1}(X_{ét}, \mathbb{Z}^c(n)) = 0 \quad \text{for } m \gg 0,$$

which means that the first $\varinjlim_m$ in the short exact sequence (1.3.15) vanishes, and we obtain isomorphisms

$$\varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n))_m, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \varinjlim_m \operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}/m\mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

It remains to note that the first $\varinjlim_m$ above is canonically isomorphic to

$$\operatorname{Hom}(H^i(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}),$$

as we observed in 0.1.2 (again, thanks to finite generation of $H^i(X_{ét}, \mathbb{Z}^c(n))$). ■

Let us summarize the results of this section.

1.3.3. Theorem. *Assuming the conjecture $\mathbf{L}^c(X_{ét}, n)$, there is a quasi-isomorphism*

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \xrightarrow{\cong} R\operatorname{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]).$$

In particular, the conjecture $\mathbf{L}^c(X_{ét}, n)$ implies that the cohomology of $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))$ is of cofinite type.

1.4 Complexes $R\widehat{\Gamma}(G_{\mathbb{R}}, (Rf_!\mathbb{Z}(n))_{\mathbb{C}})$

The duality theorem 1.3.3 deals with the complex $R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))$, so let us make a little digression to understand it. By the definition from §0.9, it sits in the distinguished triangle

$$R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n)) \to R\Gamma_c(X_{ét}, \mathbb{Z}(n)) \to R\widehat{\Gamma}(G_{\mathbb{R}}, (Rf_!\mathbb{Z}(n))_{\mathbb{C}})$$
$$\to R\widehat{\Gamma}_c(X_{ét}, \mathbb{Z}(n))[1]$$

To define cohomology with compact support, we pick a Nagata compactification

$$\begin{tikzcd}
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathfrak{X} \arrow[dl, "g"] \\
& \operatorname{Spec} \mathbb{Z} &
\end{tikzcd}$$

where j is an open immersion and g is proper. Then by definition, $Rf_! \mathbb{Z}(n) := Rg_* j_! \mathbb{Z}(n)$. As we are interested in the stalk of $Rf_! \mathbb{Z}(n)$ at $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$, let us consider the base change to $\mathbb{C}$. The schemes $f\colon X \to \operatorname{Spec} \mathbb{Z}$ and $g\colon \mathfrak{X} \to \operatorname{Spec} \mathbb{Z}$ give us $f_{\mathbb{C}}\colon X_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ and $g_{\mathbb{C}}\colon \mathfrak{X}_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$, and the open immersion $j\colon X \hookrightarrow \mathfrak{X}$ induces an open immersion $j_{\mathbb{C}}\colon X_{\mathbb{C}} \hookrightarrow \mathfrak{X}_{\mathbb{C}}$. We have the following commutative prism:

$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[rr, hook, "j_{\mathbb{C}}"] \arrow[dr, "f_{\mathbb{C}}"'] \arrow[ddd] & & \mathfrak{X}_{\mathbb{C}} \arrow[dl, "g_{\mathbb{C}}"] \arrow[ddd] \\
& \operatorname{Spec} \mathbb{C} \arrow[ddd] & \\
& & \\
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathfrak{X} \arrow[dl, "g"] \\
& \operatorname{Spec} \mathbb{Z} &
\end{tikzcd}$$

Note that the back face is also a pullback. The proper base change theorem [SGA 4, Exposé XII, Theéorème 5.1] applied to the right face of the prism (recall that the morphism g is proper) and the abelian torsion sheaf $j_! \mathbb{Z}(n)$ on $\mathfrak{X}_{ét}$, gives us an isomorphism

$$Rg_{\mathbb{C},*}(j_! \mathbb{Z}(n))_{\mathbb{C}} \cong (Rg_* j_! \mathbb{Z}(n))_{\mathbb{C}}. \tag{1.4.1}$$

Here $(j_! \mathbb{Z}(n))_{\mathbb{C}}$ denotes the inverse image of $j_! \mathbb{Z}(n)$ with respect to $\mathfrak{X}_{\mathbb{C}} \to \mathfrak{X}$, and $(Rg_* j_! \mathbb{Z}(n))_{\mathbb{C}}$ denotes the inverse image of $Rg_* j_! \mathbb{Z}(n)$ with respect to $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$. Extension by zero commutes with base change, so we have

$$(j_! \mathbb{Z}(n))_{\mathbb{C}} \cong j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}),$$

and we may rewrite (1.4.1) as

$$\underbrace{Rg_{\mathbb{C},*} j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}})}_{=:Rf_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}})} \cong (\underbrace{Rg_* j_! \mathbb{Z}(n)}_{=:Rf_! \mathbb{Z}(n)})_{\mathbb{C}}. \tag{1.4.2}$$

Now we would like to apply Artin's comparison theorem [SGA 4, Exposé XVI, Théorème 4.1]. We have the following commutative square of sites:

$$\begin{tikzcd} \mathfrak{X}_{\mathbb{C},ét} \arrow[d, "g_{\mathbb{C},ét}"'] & \mathfrak{X}_{\mathbb{C},cl} \arrow[l, "\epsilon_{\mathfrak{X}}"'] \arrow[d, "g_{\mathbb{C},cl}"] \\ (\operatorname{Spec}\mathbb{C})_{ét} & (\operatorname{Spec}\mathbb{C})_{cl} \arrow[l, "\epsilon_{\mathbb{C}}"] \end{tikzcd}$$

and for the sheaf $j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}})$, Artin's theorem gives

$$Rg_{\mathbb{C},cl,*}\epsilon_{\mathfrak{X}}^* j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}) \cong \epsilon_{\mathbb{C}}^* Rg_{\mathbb{C},ét,*} j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}).$$

Note that we have

$$\epsilon_{\mathfrak{X}}^* j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}) \cong j_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}}),$$

where ϵ_X denotes the corresponding morphism of sites $X_{\mathbb{C},cl} \to X_{\mathbb{C},ét}$. Now

$$\begin{aligned} Rf_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}}) &:= Rg_{\mathbb{C},cl,*} j_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}}) \\ &\cong \epsilon_{\mathbb{C}}^* Rg_{\mathbb{C},ét,*} j_{\mathbb{C},!}(\mathbb{Z}(n)_{\mathbb{C}}) \\ &\overset{(1.4.2)}{\cong} \epsilon_{\mathbb{C}}^*(Rg_* j_! \mathbb{Z}(n))_{\mathbb{C}} \\ &=: \epsilon_{\mathbb{C}}^*(Rf_!\mathbb{Z}(n))_{\mathbb{C}}. \end{aligned}$$

Note that $\epsilon_{\mathbb{C}}^*$ is just an equivalence of categories, and both $Rf_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}})$ and $\epsilon_{\mathbb{C}}^*(Rf_!\mathbb{Z}(n))_{\mathbb{C}}$ may be viewed as complexes of abelian groups or, more precisely, of $G_{\mathbb{R}}$-modules.

Let us calculate the sheaf $\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}})$ on $X_{\mathbb{C},cl}$. Recall that by definition,

$$\mathbb{Z}(n) := \mathbb{Q}/\mathbb{Z}(n)[-1] := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}[-1],$$

where

$$\mu_{p^r}^{\otimes n} := \underline{\operatorname{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}).$$

Base change to $X_{\mathbb{C}}$ and the inverse image ϵ_X^* commute with colimits. The sheaves $\mu_{p^r}^{\otimes n}$ become constant sheaves $\mu_{p^r}^{\otimes n}(\mathbb{C})$ on $X(\mathbb{C})$, and their colimit is given by 0.5.5.

1.4.1. Proposition. *There is an isomorphism of constant $G_{\mathbb{R}}$-equivariant sheaves on $X_{\mathbb{C},cl}$*

$$\epsilon_X^*(\mathbb{Z}(n))_{\mathbb{C}} \cong (2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1].$$

This implies that the complex $Rf_{\mathbb{C},cl,!}\epsilon_X^*(\mathbb{Z}(n)_{\mathbb{C}})$ may be identified with $R\Gamma_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1])$, and in particular, we have a quasi-isomorphism of complexes

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1])\cong R\widehat{\Gamma}(G_{\mathbb{R}},(Rf_!\mathbb{Z}(n))_{\mathbb{C}}),\tag{1.4.3}$$

where

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1]):=R\widehat{\Gamma}(G_{\mathbb{R}},R\Gamma_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1])).$$

1.4.2. Proposition. *We have a quasi-isomorphism of complexes*

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1])\cong R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}).$$

Proof. Consider the short exact sequence of $G_{\mathbb{R}}$-equivariant sheaves on $X(\mathbb{C})$

$$0\to(2\pi i)^n\,\mathbb{Z}\to(2\pi i)^n\,\mathbb{Q}\to(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}\to 0$$

which gives us a distinguished triangle

$$\begin{aligned}&R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})\to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})\\&\qquad\to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z})\to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})[1]\end{aligned}$$

and the corresponding long exact sequence in cohomology

$$\begin{aligned}\cdots\to\widehat{H}^{i-1}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})\to\widehat{H}^{i-1}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z})\to\\ \widehat{H}^{i}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})\to\widehat{H}^{i}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})\to\cdots\end{aligned}$$

Now in the spectral sequence

$$E_2^{pq}=\widehat{H}^p(G_{\mathbb{R}},H^q_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}))\Longrightarrow\widehat{H}^{p+q}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}),$$

the groups $\widehat{H}^p(G_{\mathbb{R}},H^q_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}))$ are $\mathbb{Q}$-vector spaces, and they are 2-torsion for all $p\in\mathbb{Z}$ (keep in mind that we are working with Tate cohomology). This means that $E_2^{pq}=0$ for all $p,q\in\mathbb{Z}$, and

$$\widehat{H}^i_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})=0.$$

We conclude that the morphism

$$R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1])\to R\widehat{\Gamma}_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

induces isomorphisms on cohomology. ∎

Combining the last proposition with (1.4.3), we obtain the following result.

1.4.3. Theorem. *There is a quasi-isomorphism of complexes*

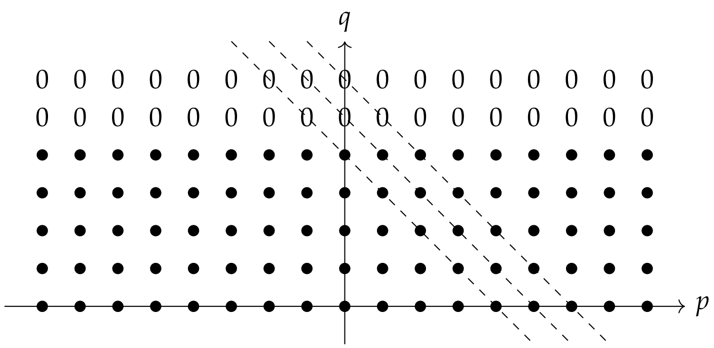
$$R\widehat{\Gamma}(G_\mathbb{R}, (Rf_!\mathbb{Z}(n))_\mathbb{C}) \cong R\widehat{\Gamma}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}).$$

The cohomology of these complexes is given by finite 2*-torsion groups.*

Proof. Tate (hyper)cohomology groups of $G_\mathbb{R}$ are always killed by $\#G_\mathbb{R} = 2$ (see 0.9.1). To see that in our case these 2-torsion groups are finite, we may consider the spectral sequence

$$E_2^{pq} = \widehat{H}^p(G_\mathbb{R}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})) \Longrightarrow \widehat{H}_c^{p+q}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}).$$

According to 0.10.1, the groups $H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ are finitely generated for all q, and they vanish for $q \gg 0$ and $q < 0$. This means that the second page of the spectral sequence looks like

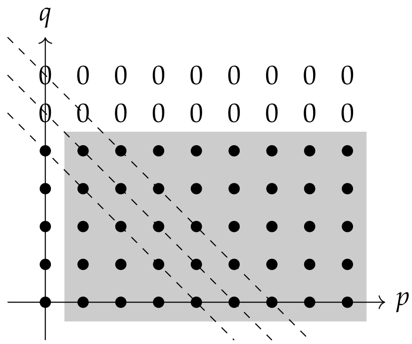


where all objects are *finite* 2-torsion. ∎

For the sake of completeness and for further reference, let us look at spectral sequences similar to the one in the last proof, but with the usual group cohomology instead of Tate cohomology. If we replace $\widehat{H}$ with H, then $H^p(G_\mathbb{R}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}))$ is not necessarily 2-torsion for $p = 0$, and the second page of the spectral sequence

$$E_2^{pq} = H^p(G_\mathbb{R}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})) \Longrightarrow H_c^{p+q}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$$

looks like



where the shaded part E_2^{pq}, $p > 0$ consists of finitely generated 2-torsion groups, the line E_2^{0q} consists of finitely generated groups, and the objects E_2^{pq} are zero for $q \gg 0$. It follows that the groups $H^i(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ are all finitely generated as well, and they are torsion for $i \gg 0$. This is in fact 2-torsion, and we may see this as follows. If $P_\bullet \twoheadrightarrow \mathbb{Z}$ is the bar-resolution of $\mathbb{Z}$ by free $\mathbb{Z}G_{\mathbb{R}}$-modules, then the morphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r] \arrow[d, "2"] & P_2 \arrow[r] \arrow[d, "2"] & P_1 \arrow[r] \arrow[d, "2"] & P_0 \arrow[r] \arrow[d, "2-N"] & 0 \\
\cdots \arrow[r] & P_3 \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{aligned}
\text{“}2\text{”}\colon P_\bullet &\to P_\bullet, \\
(2-N)\colon P_0 &\to P_0, \\
2\colon P_i &\to P_i \quad \text{for } i > 1,
\end{aligned}$$

which induces multiplication by 2 on $H^i(G, -)$ for $i > 0$ is null-homotopic [Wei1994, Theorem 6.5.8]. It is not multiplication by 2 in degree 0, but as the complex $R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ is bounded, we see that it induces multiplication by 2 on $H^i(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$ for $i \gg 0$. So we just proved the following.

1.4.4. Lemma. *The complex*

$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) = R\Gamma(G_{\mathbb{R}}, R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}))$$

is almost perfect in the sense of 0.3.3.

As for $\mathbb{Q}/\mathbb{Z}$-coefficients, we may analyze a similar spectral sequence

$$E_2^{pq} = H^p(G_{\mathbb{R}}, H_c^q(X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})) \Longrightarrow H^{p+q}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z}).$$

The second page will have groups of cofinite type on the line E_2^{0q} (see 0.10.1) and finite 2-torsion groups E_2^{pq} for $p > 0$. We have filtrations

$$\begin{aligned}(1.4.4)\quad H^{p+q} = F^0(H^{p+q}) \supseteq F^1(H^{p+q}) \supseteq F^2(H^{p+q}) \supseteq \cdots \\ \supseteq F^{p+q}(H^{p+q}) \supset F^{p+q+1}(H^{p+q}) = 0\end{aligned}$$

where

$$0 \to F^{p+1}(H^{p+q}) \to F^p(H^{p+q}) \to E_\infty^{pq} \to 0$$

Note that E_∞^{0q} will be groups of cofinite type, and E_∞^{pq} will be finite 2-torsion groups for $p > 0$, as we are going to have

$$0 \to E_{r+1}^{0q} \to E_r^{0q} \to T \to 0$$

where T is finite 2-torsion, and similarly,

$$E_{r+1}^{pq} \cong \ker d_r^{pq} / \operatorname{im} d_r^{p-r,q+r-1}$$

$$E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{pq} \xrightarrow{d_r^{pq}} E_r^{p+r,q-r+1}$$

where E_r^{pq} is finite 2-torsion for $p > 0$. It follows by induction that all the members of the filtration (1.4.4) are finite groups, except for $F^0(H^{p+q}) = H^{p+q}$ itself, which is of cofinite type, being an extension of a group of cofinite type E_∞^{0q} by a finite group $F^1(H^{p+q})$ (see 0.1.3). We also see that H^{p+q} is 2-torsion for $p+q \gg 0$. This gives us the following result.

1.4.5. Lemma. *The complex*

$$R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z}) = R\Gamma(G_\mathbb{R}, R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z}))$$

is almost of cofinite type in the sense of 0.3.7.

1.5 Complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$

1.5.1. Definition. The morphism $\alpha_{X,n}$ in $\mathbf{D}(\mathbf{Ab})$ is given by the composition of morphisms

$$\begin{aligned}R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \\ \xleftarrow{\cong} R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))\end{aligned}$$

Here the first arrow is induced by $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), -)$ and the canonical projection $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$. The second arrow is a quasi-isomorphism given by theorem 1.3.3. The third arrow is the morphism (0.9.8) from cohomology with compact support à la Milne to the usual cohomology with compact support.

Then the complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined as a cone of $\alpha_{X,n}$ in $\mathbf{D}(\mathbf{Ab})$:

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])$$

1.5.2. Remark. If $X(\mathbb{R}) = \emptyset$, then $R\widehat{\Gamma}_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is the same as $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ (see 0.9.2), so that in this case we have an isomorphism of distinguished triangles

$$\begin{tikzcd}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \arrow[r, "\mathrm{id}"] \arrow[d] & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \arrow[d] \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}[-2]) \arrow[r, "\simeq"] \arrow[d] & R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \arrow[d] \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \arrow[r, dashed, "\simeq"] \arrow[d] & R\Gamma_{fg}(X, \mathbb{Z}(n)) \arrow[d] \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \arrow[r, "\mathrm{id}"] & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1])
\end{tikzcd}$$

where the left column is the result of application of $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), -)$ to an appropriate rotation of the triangle

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Z}[1]$$

We conclude that

$$R\Gamma_{fg}(X, \mathbb{Z}(n)) \simeq R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]).$$

However, this holds only if $X(\mathbb{R}) = \emptyset$. In what follows, we are not going to make such an assumption on X, even though it would save quite some technical work. It is still helpful to keep in mind the special case $X(\mathbb{R}) = \emptyset$.

The complex of sheaves $\mathbb{Z}^c(n)$ is bounded from below, under the assumption that their cohomology groups are finitely generated (which is our conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, stated in 1.1.1).

1.5.3. Lemma. *Assuming the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, we have*

$$H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) = 0 \quad \textit{for } i < -2 \dim X.$$

Proof. The complex of sheaves $\mathbb{Z}^c(n)$ is flat, so the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

gives us a short exact sequence of étale sheaves

$$0 \to \mathbb{Z}^c(n) \to \mathbb{Q}^c(n) \to \mathbb{Q}/\mathbb{Z}^c(n) \to 0$$

and then applying $R\Gamma(X_{\acute{e}t}, -)$, we obtain a distinguished triangle in $\mathbf{D}(\mathbf{Ab})$

$$R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Q}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n))[1]$$

Now according to [Mor2014, Lemma 5.12] (note that the proof there also uses Geisser's duality), we have

$$H^i(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}^c(n)) = 0 \quad \text{for } i < -2 \dim X,$$

and the above triangle implies that

$$H^i(X_{\acute{e}t}, \mathbb{Q}^c(n)) \cong H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \quad \text{for } i < -2 \dim X.$$

However, $H^i(X_{\acute{e}t}, \mathbb{Q}^c(n))$ is a $\mathbb{Q}$-vector space, and according to the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, the groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated over $\mathbb{Z}$. This means that for $i < -2 \dim X$ these groups are trivial. ■

1.5.4. Proposition. *The complex $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect in the sense of 0.3.3, i.e. its cohomology groups $H^i_{fg}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{fg}(X, \mathbb{Z}(n)))$ are finitely generated, trivial for $i \ll 0$, and only have 2-torsion for $i \gg 0$.*

Proof. By the definition of $R\Gamma_{fg}(X, \mathbb{Z}(n))$, we have a long exact sequence in cohomology

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \arrow[r, "H^i(\alpha_{X,n})"] & H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r] & H^i_{fg}(X, \mathbb{Z}(n)) \arrow[dll, "\delta^i"] \\
& \operatorname{Hom}(H^{1-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) \arrow[r, "H^{i+1}(\alpha_{X,n})"] & H^{i+1}_c(X_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r] & \cdots
\end{tikzcd}$$

We consider short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \ker \delta^i \arrow[r] \arrow[d, equal] & H^i_{fg}(X, \mathbb{Z}(n)) \arrow[r] & \operatorname{im} \delta^i \arrow[r] \arrow[d, equal] & 0 \\
& \operatorname{coker} H^i(\alpha_{X,n}) & & \ker H^{i+1}(\alpha_{X,n}) &
\end{tikzcd}$$

By the definition of $\alpha_{X,n}$, the morphism $H^i(\alpha_{X,n})$ factors as

$$\begin{aligned}\mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}) &\to \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) \\ &\xrightarrow{\cong} \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))\end{aligned}$$

Here the morphism $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is identity, except for some *finite* 2-torsion. Indeed, this morphism sits in the long exact sequence (0.9.9):

$$\begin{aligned}\cdots \to \widehat{H}^{i-1}(G_{\mathbb{R}}, (Rf_!\mathbb{Z}(n))_{\mathbb{C}}) \to \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \\ \to \widehat{H}^i(G_{\mathbb{R}}, (Rf_!\mathbb{Z}(n))_{\mathbb{C}}) \to \cdots\end{aligned}$$

and $\widehat{H}^i(G_{\mathbb{R}}, (Rf_!\mathbb{Z}(n))_{\mathbb{C}})$ is finite 2-torsion according to 1.4.3.

The group $H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n))$ is finitely generated according to the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ (see 1.1.1). If this group is of the form $\mathbb{Z}^{\oplus r} \oplus T$, the morphism $H^i(\alpha_{X,n})$ is given by

$$\mathbb{Q}^{\oplus r} \twoheadrightarrow (\mathbb{Q}/\mathbb{Z})^{\oplus r} \rightarrowtail \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$$

where $(\mathbb{Q}/\mathbb{Z})^{\oplus r} \rightarrowtail \widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is the inclusion of the maximal divisible subgroup in the group of cofinite type

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}).$$

Both kernel and cokernel of the above map are finitely generated, hence $H^i_{fg}(X, \mathbb{Z}(n))$ is finitely generated.

As we observed in 1.5.3, again assuming the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$, we may deduce that the complex $\mathbb{Z}^c(n)$ is bounded from below. This means that for $i \ll 0$ we have

$$\ker H^{i+1}(\alpha_{X,n}) = 0, \quad H^i_{fg}(X, \mathbb{Z}(n)) \cong \operatorname{coker} H^i(\alpha_{X,n}) = H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)).$$

For $i < 1$ we have $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) = 0$, and for $i \gg 0$ we know that

$$\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \cong \mathrm{Hom}(H^{2-i}(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}) = 0,$$

again by boundedness of $\mathbb{Z}^c(n)$ from below. The only difference between $H^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ and $\widehat{H}^i_c(X_{\acute{e}t}, \mathbb{Z}(n))$ is some finite 2-torsion. ∎

1.5.5. Observation. *$R\Gamma_{fg}(X, \mathbb{Z}(n))$ is defined up to a unique isomorphism in $\mathbf{D}(\mathbf{Ab})$.*

Proof. The complex $R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2])$ consists of $\mathbb{Q}$-vector spaces, and $R\Gamma_{fg}(X, \mathbb{Z}(n))$ is almost perfect, so we are in the situation of 0.3.6. ∎

1.5.6. Observation. *Fix a distinguished triangle defining $R\Gamma_{fg}(X,\mathbb{Z}(n))$:*

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{ét},\mathbb{Z}(n)) \xrightarrow{f} R\Gamma_{fg}(X,\mathbb{Z}(n)) \\ \xrightarrow{g} R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \end{aligned}$$

1) *For each $m = 1,2,3,\ldots$ the morphism*

$$f \otimes \mathbb{Z}/m\mathbb{Z}\colon R\Gamma_c(X_{ét},\mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$$

is iso. Further, we have

$$\begin{aligned} R\Gamma_c(X_{ét},\mathbb{Z}(n)) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong R\Gamma_c(X_{ét},\mathbb{Z}/m\mathbb{Z}(n)) \\ := R\Gamma_c(X_{ét},\mathbb{Z}(n) \otimes^{\mathbf{L}} \mathbb{Z}/m\mathbb{Z}). \end{aligned}$$

2) *The morphism*

$$g \otimes \mathbb{Q}\colon R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1])$$

is iso.

Proof. The statement 1) follows from the fact that the complexes

$$R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[\ldots])$$

consist of $\mathbb{Q}$-vector spaces, and thus

$$\begin{aligned} R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[\ldots]) \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \\ \simeq R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[\ldots]) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq 0. \end{aligned}$$

Next, 2) follows from the fact that the cohomology of the étale sheaf $\mathbb{Z}(n)$ is torsion, and therefore

$$\begin{aligned} H^i(R\Gamma_c(X_{ét},\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong H^i_c(X_{ét},\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0, \\ R\Gamma_c(X_{ét},\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq 0. \end{aligned}$$

■

1.6 Complexes $R\Gamma_{W,c}(X,\mathbb{Z}(n))$

To define complexes $R\Gamma_{W,c}(X,\mathbb{Z}(n))$, we first construct a morphism

$$i_\infty^*\colon R\Gamma_{fg}(X,\mathbb{Z}(n)) \to R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}).$$

By definition, it sits in the morphism of distinguished triangles

$$
\begin{tikzcd}
R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r] \arrow[d, "\alpha_{X,n}"'] & 0 \arrow[d] \\
R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, "u_\infty^*"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[d, "\mathrm{id}"] \\
R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, dashed, "i_\infty^*"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[d] \\
R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \arrow[r] & 0
\end{tikzcd}
\tag{1.6.1}
$$

Here

$$u_\infty^*\colon R\Gamma(X_{\acute{e}t},\mathbb{Z}(n)) \to R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

is some morphism, to be defined below, such that the composition $u_\infty^*\circ\alpha_{X,n}$ is zero. Then by the axiom (TR3) there exists some morphism i_∞^*. The fact that $u_\infty^*\circ\alpha_{X,n} = 0$ will be a delicate issue, which is the main goal of this section. However, once we know that, i_∞^* is automatically unique.

1.6.1. Observation. *If i_∞^* exists, then it is unique.*

Proof of 1.6.1. We may apply 0.3.6, because $R\mathrm{Hom}(R\Gamma(X,\mathbb{Z}^c(n)),\mathbb{Q}[-2])$ is a complex of $\mathbb{Q}$-vector spaces and both

$$R\Gamma_{fg}(X,\mathbb{Z}(n)) \quad\text{and}\quad R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

are almost perfect complexes by 1.5.4 and 1.4.4. ∎

1.6.2. Proposition. *Consider the morphism*

$$\alpha^*\colon \mathbf{Sh}(X_{\acute{e}t}) \to \mathbf{Sh}(G_{\mathbb{R}},X(\mathbb{C})),$$

as described in §0.7. For the sheaf

$$\mathbb{Q}/\mathbb{Z}(n) := \bigoplus_p \varinjlim_r j_{p!}\mu_{p^r}^{\otimes n}$$

defined in §1.2 we have an isomorphism of $G_{\mathbb{R}}$-equivariant constant sheaves on $X(\mathbb{C})$

$$\alpha^*\mathbb{Q}/\mathbb{Z}(n) \cong \frac{(2\pi i)^n\,\mathbb{Q}}{(2\pi i)^n\,\mathbb{Z}} =: (2\pi i)^n\,\mathbb{Q}/\mathbb{Z}.$$

Proof. First of all, since α^* is the composition of certain inverse image functors γ^* and ϵ^* (which are left adjoint) and an equivalence of categories δ_*, the functor α^* preserves colimits, and in particular

$$\alpha^*\mathbb{Q}/\mathbb{Z}(n) \cong \bigoplus_p \varinjlim_r \alpha^* j_{p!}\mu_{p^r}^{\otimes n}. \tag{1.6.2}$$

Another formal observation is that the base change from $\operatorname{Spec}\mathbb{Z}$ to $\operatorname{Spec}\mathbb{C}$ factors through the base change to $\operatorname{Spec}\mathbb{Z}[1/p]$, and then $j_p^* \circ j_{p!} = \mathrm{id}_{\mathbf{Sh}(X[1/p]_{\acute{e}t})}$:

$$\begin{tikzcd}
\mathbf{Sh}(X[1/p]_{\acute{e}t}) \arrow[r, "j_{p!}"] \arrow[drr, "\mathrm{id}"'] & \mathbf{Sh}(X_{\acute{e}t}) \arrow[rrr, "\gamma^*"] \arrow[dr, "j_p^*"] & & & \mathbf{Sh}(X_{\mathbb{C},\acute{e}t}) \\
& & \mathbf{Sh}(X[1/p]_{\acute{e}t}) \arrow[urr, dashed] & &
\end{tikzcd}$$

which means that we may safely erase "$j_{p!}$" in (1.6.2), and everything boils down to calculating the sheaves

$$\alpha^*\mu_{p^r}^{\otimes n} = \alpha^*\underline{\mathrm{Hom}}_{X[1/p]}(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}).$$

As we base change to $\operatorname{Spec}\mathbb{C}$, the étale sheaf μ_{p^r} simply becomes the constant sheaf $\mu_{p^r}(\mathbb{C})$ on $X(\mathbb{C})$, and

$$\alpha^*\mu_{p^r}^{\otimes n} = \underline{\mathrm{Hom}}_{X(\mathbb{C})}(\mu_{p^r}^{\otimes(-n)}(\mathbb{C}), \mathbb{Z}/p^r\mathbb{Z}).$$

In 0.5.5 we calculated the colimit of such things to be $(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}$. ∎

1.6.3. Definition. The morphism

$$u_\infty^*\colon R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})$$

is given by the composition

$$\begin{aligned} &R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) := R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n))[-1] \\ &\xrightarrow{v_\infty^*[-1]} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})[-1] \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \end{aligned}$$

Here the last arrow is induced by $(2\pi i)^n\,\mathbb{Q}/\mathbb{Z}[-1] \to (2\pi i)^n\,\mathbb{Z}$, which comes from the distinguished triangle of constant $G_\mathbb{R}$-equivariant sheaves

$$(2\pi i)^n\,\mathbb{Z} \to (2\pi i)^n\,\mathbb{Q} \to (2\pi i)^n\,\mathbb{Q}/\mathbb{Z} \to (2\pi i)^n\,\mathbb{Z}[1]$$

and the arrow

$$v_\infty^*\colon R\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})$$

is induced by the morphism

$$\Gamma_c(X_{\acute{e}t}, \mathbb{Q}/\mathbb{Z}(n)) \to \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), \alpha^*\mathbb{Q}/\mathbb{Z}(n)) \cong \Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Q}/\mathbb{Z})$$

(see 0.8.3 and 1.6.2).

1.6.4. Theorem. *For any arithmetic scheme X one has $u_\infty^* \circ \alpha_{X,n} = 0$ in the derived category.*

This seems to be rather nontrivial; our proof will be based on the following result about ℓ-adic cohomology.

1.6.5. Proposition. *Let $f\colon X \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic scheme (that is, with f separated, of finite type). Let $n < 0$. Then for any prime ℓ we have*

$$(H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}})_{div} = 0.$$

Proof. Let us recall some facts about ℓ-adic cohomology. We refer to [SGA 5, Exposé VI] for details. Let us first consider the sheaf $\mathbb{Z}_\ell(n)$. It is a **constructible $\mathbb{Z}_\ell$-sheaf*** on X in the sense of [SGA 5, Exposé VI, 1.1.1]. We would like to compare the cohomology of $\mathbb{Z}_\ell(n)$ on $X_{\overline{\mathbb{Q}},\acute{e}t}$ and $X_{\overline{\mathbb{F}_p},\acute{e}t}$, where p is some prime different from ℓ, to be determined later. For this we fix some algebraic closures $\overline{\mathbb{Q}}/\mathbb{Q}$ and $\overline{\mathbb{F}_p}/\mathbb{F}_p$ and consider the corresponding morphisms

$$\overline{\eta}\colon \operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\mathbb{Z}, \quad \overline{x}\colon \operatorname{Spec}\overline{\mathbb{F}_p} \to \operatorname{Spec}\mathbb{Z}.$$

Let $X_{\overline{\mathbb{Q}},\acute{e}t}$ and $X_{\overline{\mathbb{F}_p},\acute{e}t}$ be the pullbacks of X along the above morphisms:

$$\begin{tikzcd}
X_{\overline{\mathbb{Q}}} \arrow[r] \arrow[d, "f_{\overline{\mathbb{Q}}}"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] & X_{\overline{\mathbb{F}_p}} \arrow[l] \arrow[d, "f_{\overline{\mathbb{F}_p}}"] \arrow[dl, phantom, "\llcorner"] \\
\operatorname{Spec}\overline{\mathbb{Q}} \arrow[r, "\overline{\eta}"'] & \operatorname{Spec}\mathbb{Z} & \operatorname{Spec}\overline{\mathbb{F}_p} \arrow[l, "\overline{x}"]
\end{tikzcd}$$

According to [SGA 5, Exposé VI, 2.2.3], the proper base change theorem holds for constructible $\mathbb{Z}_\ell$-sheaves. It gives us isomorphisms

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t}, \mathbb{Z}_\ell(n)) \cong (R^i f_! \mathbb{Z}_\ell(n))_{\overline{\eta}}, \quad H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t}, \mathbb{Z}_\ell(n)) \cong (R^i f_! \mathbb{Z}_\ell(n))_{\overline{x}},$$

where $R^i f_! \mathbb{Z}_\ell(n)$ is the same sheaf on $\operatorname{Spec}\mathbb{Z}$, and we take its different stalks to get cohomology with compact support on different fibers. The construction of higher direct images with proper support $R^i f_! \mathcal{F}$ for ℓ-adic sheaves is given in [SGA 5, Exposé VI, §2.2]. The key nontrivial fact that we need

*Or simply $\mathbb{Z}_\ell$**-sheaf** in the terminology of [SGA $4\frac{1}{2}$, Rapport].

is that for every morphism (of locally noetherian schemes) $f\colon X \to Y$, separated of finite type, if $\mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf on X, then $R^i f_! \mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf on Y.

According to [SGA 5, Exposé VI, 1.2.6], for a projective system of abelian sheaves $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ on $X_{ét}$, the following are equivalent:

1) $\mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf,

2) every open subscheme $U \subset X$ is a finite union of locally closed pieces Z_i where $\mathcal{F}|_{Z_i}$ is a **twisted constant constructible $\mathbb{Z}_\ell$-sheaf**[*].

Being "twisted constant" means that each sheaf $\mathcal{F}_n$ in the projective system $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is locally constant. The importance of twisted constant sheaves is explained by the following property [SGA 5, Exposé VI, 1.2.4, 1.2.5]: for a connected locally noetherian scheme X, the category of twisted constant $\mathbb{Z}_\ell$-constructible sheaves on X is equivalent to the category of finitely generated $\mathbb{Z}_\ell$-modules with a continuous action of the étale fundamental group $\pi_1^{ét}(X)$.

In our setting, all this means that there exists an open subscheme

$$U = \operatorname{Spec} \mathbb{Z}_S \subset \operatorname{Spec} \mathbb{Z},$$

where $\mathbb{Z}_S$ denotes the localization of $\mathbb{Z}$ at a finite set of primes S, such that the sheaves $R^i f_! \mathbb{Z}_\ell(n)$ are twisted constant on U. By removing all the necessary bad primes, we can make sure this holds for all i.

Now according to [Elements, Book IX, Proposition 20], there exists some prime $p \notin S$ (that is, $(p) \in U$), for which we may consider the following picture:

$$\begin{tikzcd}
X_{\overline{\mathbb{Q}}} \arrow[r] \arrow[d, "f_{\overline{\mathbb{Q}}}"'] \arrow[dr, phantom, "\lrcorner"] & X_U \arrow[d, "f_U"] \arrow[dr, phantom, "\llcorner"] & X_{\overline{\mathbb{F}_p}} \arrow[l] \arrow[d, "f_{\overline{\mathbb{F}_p}}"] \\
\operatorname{Spec} \overline{\mathbb{Q}} \arrow[r, "\overline{\eta}"'] & U & \operatorname{Spec} \overline{\mathbb{F}_p} \arrow[l, "\overline{x}"]
\end{tikzcd}$$

It follows that we have isomorphisms

(1.6.3)
$$H^i_c(X_{\overline{\mathbb{Q}},ét}, \mathbb{Z}_\ell(n)) \cong (R^i f_{U,!} \mathbb{Z}_\ell(n))_{\overline{\eta}} \cong (R^i f_{U,!} \mathbb{Z}_\ell(n))_{\overline{x}} \cong H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n)),$$

of finitely generated $\mathbb{Z}_\ell$-modules with continuous action of

$$\pi_1^{ét}(U) \cong \operatorname{Gal}(\mathbb{Q}_S/\mathbb{Q}),$$

where $\mathbb{Q}_S/\mathbb{Q}$ denotes a maximal extension of $\mathbb{Q}$ unramified outside of S. We note that $(R^i f_{U,!} \mathbb{Z}_\ell(n))_{\overline{\eta}}$ naturally carries an action of $\pi_1^{ét}(U, \overline{\eta})$, while

[*] A **faisceau lisse** in the terminology of [SGA $4\frac{1}{2}$, Rapport].

$(R^i f_{U,!}\mathbb{Z}_\ell(n))_{\overline{x}}$ carries an action of $\pi_1^{\acute{e}t}(U,\overline{x})$, and the isomorphism in the middle of (1.6.3) sweeps under the rug an identification of $\pi_1^{\acute{e}t}(U,\overline{\eta})$ with $\pi_1^{\acute{e}t}(U,\overline{x})$.

To state this more accurately, note that the $\mathbb{Z}_\ell$-module $H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n))$ carries a natural action of $G_{\mathbb{Q}}$ while $H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))$ carries a natural action of $G_{\mathbb{F}_p}$. After making the necessary choices, we have $G_{\mathbb{Q}_p}\subset G_{\mathbb{Q}}$ and a short exact sequence

$$1\to I_p\to G_{\mathbb{Q}_p}\to G_{\mathbb{F}_p}\to 1$$

where I_p is the inertia subgroup, acting trivially on $H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n))$. We have thus isomorphisms of finitely generated $\mathbb{Z}_\ell$-modules

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n))\cong H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n)),$$

equivariant under the action of $G_{\mathbb{Q}_p}/I_p$ on the left hand side and of $G_{\mathbb{F}_p}$ on the right hand side. To relate all this to $\mathbb{Q}_\ell(n)$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$-coefficients, note that we have the following isomorphic long exact sequences in cohomology

(1.6.4)
$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{i-1}_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \arrow[r, "\cong"] \arrow[d, "\delta"] & H^{i-1}_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \arrow[d, "\delta"] \\
H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n)) \arrow[r, "\cong"] \arrow[d, "\phi"] & H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n)) \arrow[d, "\phi"] \\
H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell(n)) \arrow[r, "\cong"] \arrow[d, "\psi"] & H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell(n)) \arrow[d, "\psi"] \\
H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \arrow[r] \arrow[d, "\cong"] & H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Here

$$H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Q}_\ell(n))=H^i_c(X_{\overline{\mathbb{Q}},\acute{e}t},\mathbb{Z}_\ell(n))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell,$$
$$H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Q}_\ell(n))=H^i_c(X_{\overline{\mathbb{F}_p},\acute{e}t},\mathbb{Z}_\ell(n))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell,$$

and the arrows ϕ above are canonical localization morphisms. The horizontal arrows are equivariant isomorphisms in the above sense. Note that we have

$$\begin{aligned} H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}} \rightarrowtail H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{Q}_p}/I_p} \\ \cong H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}}, \end{aligned}$$

so in order to prove that

$$(H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}})_{div} = 0,$$

it will be enough to show that

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = 0.$$

From now on we move to the characteristic p and consider the fixed points of $G_{\mathbb{F}_p}$ acting on the $\mathbb{Z}_\ell$-module $H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. In the long exact sequence (1.6.4), we have (keeping in mind that ϕ is merely the localization morphism):

$$\begin{aligned} \ker\phi &= H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))_{tor}, \\ \ker\psi &= \operatorname{im}\phi \cong H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))/\ker\phi \\ &= \frac{H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))}{H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))_{tor}} =: H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))_{cotor}, \\ \operatorname{im}\psi &= H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div}. \end{aligned}$$

This gives us a short exact sequence

$$\begin{aligned} 0 \to H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))_{cotor} \to H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell(n)) \\ \to H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div} \to 0 \end{aligned}$$

After taking the $G_{\mathbb{F}_p}$-invariants, we obtain a long exact sequence of cohomology groups

$$\begin{aligned} (1.6.5) \quad 0 \to (H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))_{cotor})^{G_{\mathbb{F}_p}} \to H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}} \\ \to (H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \to H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Z}_\ell(n))_{cotor}) \to \cdots \end{aligned}$$

We claim that

$$(1.6.6) \qquad H^i_c(X_{\overline{\mathbb{F}_p},ét},\mathbb{Q}_\ell(n))^{G_{\mathbb{F}_p}} = 0.$$

Indeed, according to [SGA 7, Exposé XXI, 5.5.3], the eigenvalues of the geometric Frobenius acting on $H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell)$ are algebraic integers. We are twisting $\mathbb{Q}_\ell$ by n, so the eigenvalues of Frobenius lie in $p^{-n}\,\overline{\mathbb{Z}}$. Since $n < 0$ by our assumption, this implies that 1 does not occur as an eigenvalue.

Now (1.6.6) and the long exact sequence (1.6.5) imply that there is a monomorphism

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}} \rightarrowtail H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor}),$$

which restricts to a monomorphism between the maximal divisible subgroups

$$((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} \rightarrowtail H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})_{div}.$$

However, $H^1(G_{\mathbb{F}_p}, H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Z}_\ell(n))_{cotor})$ is a finitely generated $\mathbb{Z}_\ell$-module, and therefore its maximal divisible subgroup is trivial. We have therefore

$$(H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_{\mathbb{F}_p}})_{div} = ((H^i_c(X_{\overline{\mathbb{F}_p},ét}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{div})^{G_{\mathbb{F}_p}})_{div} = 0.$$

(For the first equality, note that for any G-module A one has $((A_{div})^G)_{div} = (A^G)_{div}$.) ■

Now we are ready to prove 1.6.4. The morphism $\alpha_{X,n}$ is defined on

$$R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]),$$

which is a complex of $\mathbb{Q}$-vector spaces, so it will be enough to show that v_∞^* is a torsion element in the abelian group

$$\mathrm{Hom}_{\mathbf{D}(\mathbf{Ab})}(R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)),\ R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})).$$

The complexes $R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))$ and $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})$ are almost of cofinite type in the sense of 0.3.7. Indeed, we observed it in 1.4.5 for $R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})$, and for $R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n))$, by the duality theorem 1.3 we have

$$\begin{aligned} H^i_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) = H^{i-1}_c(X_{ét}, \mathbb{Z}(n)) &\overset{\text{up to 2-torsion}}{\approx} \widehat{H}^{i-1}_c(X_{ét}, \mathbb{Z}(n)) \\ &\cong \mathrm{Hom}(H^{3-i}(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}/\mathbb{Z}(n)) \end{aligned}$$

and the groups $H^{3-i}(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated by our conjecture $\mathbf{L}^c(X_{ét}, n)$ (see 1.1.1), trivial for $i \ll 0$ by 1.5.3 (again, assuming $\mathbf{L}^c(X_{ét}, n)$) and finite 2-torsion for $i \gg 0$. Therefore, according to 0.3.8, to show that $v_\infty^*\colon R\Gamma_c(X_{ét}, \mathbb{Q}/\mathbb{Z}(n)) \to R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}/\mathbb{Z})$ is torsion in $\mathbf{D}(\mathbf{Ab})$, it is enough to show that the corresponding morphisms on the maximal divisible subgroups

$$H^i_c(v^*_\infty)_{div}\colon H^i_c(X_{ét},\mathbb{Q}/\mathbb{Z}(n))_{div}\to H^i_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z})_{div}$$

are all trivial.

The morphism $H^i_c(v^*_\infty)$ factors through $H^i_c(X_{\overline{\mathbb{Q}},ét},\mu^{\otimes n})^{G_\mathbb{Q}}$, where $\mu^{\otimes n}$ is the sheaf of all roots of unity on $X_{\overline{\mathbb{Q}},ét}$ twisted by n. We have therefore

$$\begin{tikzcd}
H^i_c(X_{ét},\mathbb{Q}/\mathbb{Z}(n))_{div} \arrow[rr, "H^i_c(v^*_\infty)_{div}"] \arrow[dr, dashed] & & H^i_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}/\mathbb{Z})_{div} \\
& \left(H^i_c(X_{\overline{\mathbb{Q}},ét},\mu^{\otimes n})^{G_\mathbb{Q}}\right)_{div} \arrow[ur, dashed] &
\end{tikzcd}$$

Now

$$\left(H^i_c(X_{\overline{\mathbb{Q}},ét},\mu^{\otimes n})^{G_\mathbb{Q}}\right)_{div}\cong\left(\bigoplus_\ell H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}}\right)_{div}\cong$$
$$\bigoplus_\ell\left(H^i_c(X_{\overline{\mathbb{Q}},ét},\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_\mathbb{Q}}\right)_{div},$$

where all summands are trivial according to 1.6.5. ■

1.6.6. Corollary. *The morphism i^*_∞ is torsion in the derived category, i.e. $i^*_\infty\otimes\mathbb{Q}=0$.*

Proof. Let us examine the morphism of distinguished triangles (1.6.1) that defines i^*_∞; in particular, the commutative diagram

$$\begin{tikzcd}
R\Gamma_c(X_{ét},\mathbb{Z}(n)) \arrow[r] \arrow[d, "u^*_\infty"'] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[dl, "i^*_\infty"] \\
R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) &
\end{tikzcd}$$

According to 0.3.6, the morphism

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(R\Gamma_{fg}(X,\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))\to$$
$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(R\Gamma_c(X_{ét},\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))$$

induced by the composition with $R\Gamma_c(X_{ét},\mathbb{Z}(n))\to R\Gamma_{fg}(X,\mathbb{Z}(n))$, is mono, and therefore

$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(R\Gamma_{fg}(X,\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))\otimes_\mathbb{Z}\mathbb{Q}\to$$
$$\operatorname{Hom}_{\mathbf{D}(\mathbf{Ab})}(R\Gamma_c(X_{ét},\mathbb{Z}(n)),R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))\otimes_\mathbb{Z}\mathbb{Q}$$

is mono as well. However, we just saw in the proof of 1.6.4 that $u_\infty^* \otimes \mathbb{Q} = 0$, and this implies that $i_\infty^* \otimes \mathbb{Q} = 0$. ■

Now that we know that i_∞^* exists (and is unique), we are ready to define Weil-étale complexes.

1.6.7. Definition. $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is an object in the derived category $\mathbf{D}(\mathbf{Ab})$ which is a mapping fiber of i_∞^*:

$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\ \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1]$$

The **Weil-étale cohomology with compact support** is given by

$$H^i_{W,c}(X, \mathbb{Z}(n)) := H^i(R\Gamma_{W,c}(X, \mathbb{Z}(n))).$$

Note that this defines $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ up to a non-unique isomorphism in $\mathbf{D}(\mathbf{Ab})$, and the groups $H^i_{W,c}(X, \mathbb{Z}(n))$ are also defined up to a non-unique isomorphism.

1.6.8. Proposition. *The conjecture* $\mathbf{L}^c(X_{\acute{e}t}, n)$ *implies that* $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ *is a perfect complex.*

Proof. By definition, we have a long exact sequence in cohomology

$$\cdots \to H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \to H^i_{W,c}(X, \mathbb{Z}(n)) \to \\ H^i_{fg}(X, \mathbb{Z}(n)) \xrightarrow{H^i(i_\infty^*)} H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \to \cdots$$

The groups $H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})$ and $H^i_{fg}(X, \mathbb{Z}(n))$ are finitely generated by 1.4.4 and 1.5.4. They vanish for $i \ll 0$, but they are finite 2-torsion for $i \gg 0$. I claim that $H^i(i_\infty^*)$ is an isomorphism for $i \gg 0$, meaning that this 2-torsion in higher degrees does not appear in $H^i_{W,c}(X, \mathbb{Z}(n))$. We have a commutative diagram

$$\begin{tikzcd}
H^i_c(X_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r] \arrow[d, "H^i(u_\infty^*)"'] & H^i_{fg}(X, \mathbb{Z}(n)) \arrow[dl, "H^i(i_\infty^*)"] \\
H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) &
\end{tikzcd}$$

The morphism $H^i(u_\infty^*)$ is iso for $i \gg 0$, hence $H^i(i_\infty^*)$ is surjective for $i \gg 0$. However, $H^i_{fg}(X, \mathbb{Z}(n))$ and $H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z})$ have the same 2-torsion for $i \gg 0$,and $H^i(i_\infty^*)$ is iso for $i \gg 0$.

■

1.6.9. Proposition. *The determinant* $\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$ *is well-defined up to a canonical isomorphism.*

Proof. For two different choices of a mapping fiber of i_∞^*, we obtain an isomorphism of distinguished triangles

$$\begin{tikzcd}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r, dashed, "\cong"] \arrow[d] & R\Gamma_{W,c}(X,\mathbb{Z}(n))' \arrow[d] \\
R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "\mathrm{id}"] \arrow[d, "i_\infty^*"] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[d, "i_\infty^*"] \\
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[r, "\mathrm{id}"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[d] \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \arrow[r, dashed, "\cong"] & R\Gamma_{W,c}(X,\mathbb{Z}(n))'[1]
\end{tikzcd}$$

Here the dashed arrows are not canonical, but this does not affect the determinants, because these are functorial with respect to isomorphisms of triangles (see 0.4.1). The only technical issue is that the complexes $R\Gamma_{fg}(X,\mathbb{Z}(n))$ and $R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$ may have unbounded 2-torsion, unless $X(\mathbb{R}) = \emptyset$. However, we know that the arrow

$$H^i(i_\infty^*)\colon H^i_{fg}(X,\mathbb{Z}(n)) \to H^i_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

is an isomorphism for $i \gg 0$. Therefore, taking the truncations $\tau_{\le m}$ for m big enough, we obtain a commutative diagram where the columns still induce long exact sequences in cohomology:

$$\begin{tikzcd}
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r, dashed, "\cong"] \arrow[d] & R\Gamma_{W,c}(X,\mathbb{Z}(n))' \arrow[d] \\
\tau_{\le m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, "\mathrm{id}"] \arrow[d, "i_\infty^*"] & \tau_{\le m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[d, "i_\infty^*"] \\
\tau_{\le m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[r, "\mathrm{id}"] \arrow[d] & \tau_{\le m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[d] \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \arrow[r, dashed, "\cong"] & R\Gamma_{W,c}(X,\mathbb{Z}(n))'[1]
\end{tikzcd}$$

which gives us the desired canonical isomorphism

$$\begin{aligned}
&\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \\
&\quad \det_{\mathbb{Z}} \tau_{\le m}R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} (\det_{\mathbb{Z}} \tau_{\le m}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))^{-1} \\
&\qquad\qquad \cong \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))'.
\end{aligned}$$

■

1.6.10. Remark. Our methods establish existence of i_∞^* only as a morphism in the derived category and $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is defined only up to a non-canonical quasi-isomorphism. It is probably possible to construct i_∞^* as a canonical morphism in the category of complexes. This would give us a canonical construction of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ as a complex. Another possibility to make things canonical is to work with the derived ∞-category [Lur2006].

The reader will note that the non-canonicity of $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ in the present construction is not only aesthetically unpleasant, but will also give us some technical troubles later on, for instance in §1.8.

1.7 Splitting of $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}$

The following result will be crucial in the next chapter.

1.7.1. Proposition. *There is a direct sum decomposition*

$$\begin{aligned} &R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \\ &\qquad R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1]. \end{aligned}$$

This isomorphism is not canonical, but induces a canonical isomorphism

$$\begin{aligned} &(\det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \det\nolimits_{\mathbb{Q}}(R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &\cong \det\nolimits_{\mathbb{Q}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \otimes_{\mathbb{Q}} \det\nolimits_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1]. \end{aligned}$$

Proof. Everything has to do with the cohomology of $R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n))$ and the morphism i_∞^* being torsion. In fact we already noted in 1.5.6 that the distinguished triangle defining $R\Gamma_{fg}(X, \mathbb{Z}(n))$

$$\begin{aligned} &R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \xrightarrow{\alpha_{X,n}} R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \\ &\qquad\qquad \xrightarrow{g} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \end{aligned}$$

after tensoring with $\mathbb{Q}$ gives us an isomorphism

$$g \otimes \mathbb{Q} \colon R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]).$$

Now examine the triangle that defines $R\Gamma_{W,c}(X, \mathbb{Z}(n))$:

$$\begin{aligned} &R\Gamma_{W,c}(X, \mathbb{Z}(n)) \xrightarrow{h} R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}) \\ &\qquad\qquad \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \end{aligned}$$

According to 1.6.6, the morphism i_∞^* is torsion, so that $i_\infty^*\otimes\mathbb{Q}=0$ and tensoring with $\mathbb{Q}$ gives a distinguished triangle

$$\begin{aligned}R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Q}&\xrightarrow{h\otimes\mathbb{Q}} R\Gamma_{fg}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Q}\\ &\xrightarrow{0} R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})\to R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes_{\mathbb{Z}}\mathbb{Q}[1]\end{aligned}$$

To shorten the notation, let us write $[-,-]$ instead of $R\mathrm{Hom}(-,-)$ and $(-)_{\mathbb{Q}}$ instead of $-\otimes_{\mathbb{Z}}\mathbb{Q}$. We have an isomorphism of distinguished triangles

(1.7.1)

$$\begin{tikzcd}
R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}} \arrow[r, "\mathrm{id}"] \arrow[d, "h\otimes\mathbb{Q}"] & R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}} \arrow[r, dashed, "\cong"] \arrow[d, "(h\circ g)\otimes\mathbb{Q}"] & {\begin{array}{c}[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]\\ \oplus\\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})[-1]\end{array}} \arrow[d] \\
R\Gamma_{fg}(X,\mathbb{Z}(n))_{\mathbb{Q}} \arrow[r, "g\otimes\mathbb{Q}", "\cong"'] \arrow[d, "0"] & {[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[r, "\mathrm{id}"] \arrow[d, "0"] & {[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[d] \\
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \arrow[r, "\mathrm{id}"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \arrow[r, "\mathrm{id}"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q}) \arrow[d] \\
R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}}[1] \arrow[r, "\mathrm{id}"] & R\Gamma_{W,c}(X,\mathbb{Z}(n))_{\mathbb{Q}}[1] \arrow[r, dashed, "\cong"] & {\begin{array}{c}[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}]\\ \oplus\\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})\end{array}}
\end{tikzcd}$$

Here the right triangle is distinguished, being the direct sum of the distinguished triangles

$$\begin{aligned}R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1])&\xrightarrow{\mathrm{id}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1])\\ &\to 0\to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q})\end{aligned}$$

and

$$\begin{aligned}R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})[-1]\to 0\to R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})&\\ \xrightarrow{\mathrm{id}} R\Gamma_c(G_{\mathbb{Q}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Q})&\end{aligned}$$

The two dashed arrows in (1.7.1) exist thanks to the axiom (TR3), and they are isomorphisms by the triangulated 5-lemma. We note that these arrows are by no means unique*. To see that the obtained splitting is canonical on

*It is a well-known lemma that in a triangulated category, a distinguished triangle $X\xrightarrow{u} Y\xrightarrow{v} Z\xrightarrow{w} X[1]$ splits whenever one of the morphisms u,v,w is zero—[Verdier-thèse, Chapitre II, Corollaire 1.2.6]. I basically recalled the proof for our case to stress that such a splitting is not canonical.

the level of determinants, we argue as in 1.6.9. The isomorphism of triangles

$$\begin{tikzcd}
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \arrow[r, "\mathrm{id}"] \arrow[d] & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \arrow[d] \\
R\Gamma_{W,c}(X, \mathbb{Z}(n))_{\mathbb{Q}} \arrow[r, "f", "\cong"'] \arrow[d] & \begin{array}{c}[R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1]\end{array} \arrow[d] \\
R\Gamma_{fg}(X, \mathbb{Z}(n))_{\mathbb{Q}} \arrow[r, "g\otimes\mathbb{Q}", "\cong"'] \arrow[d, "0"] & [R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]] \arrow[d] \\
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q}) \arrow[r, "\mathrm{id}"] & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})
\end{tikzcd}$$

induces by 0.4.1 a commutative diagram

$$\begin{tikzcd}
\begin{array}{c}\det_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \\ \otimes_{\mathbb{Q}} \\ \det_{\mathbb{Q}} R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}\end{array} \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\otimes\det(g\otimes\mathbb{Q})"', "\cong"] & \det_{\mathbb{Q}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, "\cong"', "\det(f)"] \\
\begin{array}{c}\det_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \\ \otimes_{\mathbb{Q}} \\ \det_{\mathbb{Q}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1]\end{array} \arrow[r, "\cong"] & \det_{\mathbb{Q}} \begin{pmatrix} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \end{pmatrix}
\end{tikzcd}$$

Here the top arrow is canonical, and the left arrow as well; composing them, we obtain a canonical isomorphism

$$\det_{\mathbb{Q}}(R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong$$
$$\det_{\mathbb{Q}} R\Gamma_c(G_{\mathbb{Q}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1] \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1].$$

■

1.7.2. Remark. This means that for the Weil-étale cohomology with rational coefficients, we could take as the definition

$$R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Q})[-1],$$

which would simplify things a lot. However, it is crucial for us to work with $R\Gamma_{W,c}(X, \mathbb{Z}(n))$. In the next chapter, this will mean that we will state conjectures about special values of $\zeta(X,s)$ up to a sign ± 1 and not merely up to a multiplier $x \in \mathbb{Q}^\times$. Of course the latter would be much easier.

1.8 Compatibilities with open-closed decompositions

We say that we have an **open-closed decomposition** of a scheme X if there are given morphisms

$$U \hookrightarrow X \leftarrow Z$$

where $U \hookrightarrow X$ is an inclusion of an open subscheme of X and $Z \to X$ is a closed immersion where $Z = X \setminus U$. The goal of this section is to prove the following result.

1.8.1. Proposition. *An open-closed decomposition of arithmetic schemes*

$$U \hookrightarrow X \leftarrow Z$$

induces a canonical isomorphism

$$\det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \cong \det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(U, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(Z, \mathbb{Z}(n)). \tag{1.8.1}$$

Morally, an open-closed decomposition should induce a distinguished triangle of Weil-étale complexes

$$\begin{aligned} R\Gamma_{W,c}(U, \mathbb{Z}(n)) \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{W,c}(Z, \mathbb{Z}(n)) \\ \to R\Gamma_{W,c}(U, \mathbb{Z}(n))[1] \end{aligned} \tag{1.8.2}$$

and the corresponding long exact sequence in cohomology

$$\begin{aligned} \cdots \to H^i_{W,c}(U, \mathbb{Z}(n)) \to H^i_{W,c}(X, \mathbb{Z}(n)) \to H^i_{W,c}(Z, \mathbb{Z}(n)) \\ \to H^{i+1}_{W,c}(U, \mathbb{Z}(n)) \to \cdots \end{aligned}$$

However, with the definition of $R\Gamma_{W,c}(-, \mathbb{Z}(n))$ that we have at the moment, obtaining such a distinguished triangle seems to be a nontrivial task, and even the complexes in (1.8.2) are defined only up to a non-unique isomorphism in the derived category.

1.8.2. Remark. The main technical issue is the following. Given a morphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd} \tag{1.8.3}$$

sometimes it is tempting to consider its “cone”, i.e. complete the above diagram to a 3×3-diagram with distinguished rows and columns

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(ac)}"] & X''[1] \arrow[d] \\
X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2]
\end{tikzcd}$$

where all squares commute, except for the bottom right square, which anti-commutes*. Whenever it is possible, Neeman in [Nee1991] says that (1.8.3) is **middling good**. Unfortunately, not every morphism of triangles is middling good (see [Nee1991, Example 2.6]). It seems like the best result one can obtain in general is that given a diagram with distinguished rows

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

there exists *some* morphism $Z \to Z'$ making the above diagram into a middling good morphism of triangles (this is done using the axiom (TR4); see e.g. [BBD1982, Proposition 1.1.11] or [May2001, Lemma 2.6]).

The reader may consult [Nee1991] for a thorough discussion of this issue. The bottom line is that we should be careful and never expect an arbitrary morphism of distinguished triangles to be completed to a 3×3-diagram.

*The anti-commutativity comes from the following sign issue. The rotation axiom (TR2) says that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished. So for a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, its full rotation by 1 is not $X[1] \xrightarrow{u[1]} Y[1] \xrightarrow{v[1]} Z[1] \xrightarrow{w[1]} X[2]$ but rather $X[1] \xrightarrow{-u[1]} Y[1] \xrightarrow{-v[1]} Z[1] \xrightarrow{-w[1]} X[2]$. The latter is isomorphic to, say, $X[1] \xrightarrow{u[1]} Y[1] \xrightarrow{v[1]} Z[1] \xrightarrow{-w[1]} X[2]$, so we just have to put a minus sign somewhere. The usual convention is that in the 3×3-diagram, the bottom right square anti-commutes.

$R\Gamma_{fg}(X,\mathbb{Z}(n))$ and open-closed decompositions

For an open-closed decomposition $U \hookrightarrow X \leftarrow Z$, the cohomology of $\mathbb{Z}^c(n)$ gives a distinguished triangle

$$R\Gamma(Z_{ét},\mathbb{Z}^c(n)) \to R\Gamma(X_{ét},\mathbb{Z}^c(n)) \to R\Gamma(U_{ét},\mathbb{Z}^c(n)) \to R\Gamma(Z_{ét},\mathbb{Z}^c(n))[1]$$

(see 0.11.1). Applying to it $R\mathrm{Hom}(-,\mathbb{Q}[-2])$, we obtain a distinguished triangle

$$(1.8.4)\quad \begin{aligned} &R\mathrm{Hom}(R\Gamma(U_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \to R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \\ &\to R\mathrm{Hom}(R\Gamma(Z_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \to R\mathrm{Hom}(R\Gamma(U_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \end{aligned}$$

Similarly, for étale cohomology with compact support, we have a distinguished triangle
(1.8.5)

$$R\Gamma_c(U_{ét},\mathbb{Z}(n)) \to R\Gamma_c(X_{ét},\mathbb{Z}(n)) \to R\Gamma_c(Z_{ét},\mathbb{Z}(n)) \to R\Gamma_c(U_{ét},\mathbb{Z}(n))[1]$$

Then one can to check that $(\alpha_{U,n},\alpha_{X,n},\alpha_{Z,n})$ give a morphism of triangles (1.8.4) and (1.8.5).

1.8.3. Lemma. *We have the following commutative diagram in the derived category:*

$$(1.8.6)\quad \begin{tikzcd} R\mathrm{Hom}(R\Gamma(U_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r, "\alpha_{U,n}"] \arrow[d] & R\Gamma_c(U_{ét},\mathbb{Z}(n)) \arrow[d] \\ R\mathrm{Hom}(R\Gamma(X_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r, "\alpha_{X,n}"] \arrow[d] & R\Gamma_c(X_{ét},\mathbb{Z}(n)) \arrow[d] \\ R\mathrm{Hom}(R\Gamma(Z_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-2]) \arrow[r, "\alpha_{Z,n}"] \arrow[d] & R\Gamma_c(Z_{ét},\mathbb{Z}(n)) \arrow[d] \\ R\mathrm{Hom}(R\Gamma(U_{ét},\mathbb{Z}^c(n)),\mathbb{Q}[-1]) \arrow[r, "\alpha_{U,n}[1]"] & R\Gamma_c(U_{ét},\mathbb{Z}(n))[1] \end{tikzcd}$$

Now in the diagram (1.8.6) we may pick a cone of each arrow $\alpha_{U,n}$, $\alpha_{X,n}$, $\alpha_{Z,n}$, which is by definition $R\Gamma_{fg}(U,\mathbb{Z}(n))$, $R\Gamma_{fg}(X,\mathbb{Z}(n))$, $R\Gamma_{fg}(Z,\mathbb{Z}(n))$ respectively. According to 1.5.5, each of these is defined up to a unique isomorphism in the derived category.

$$\begin{tikzcd}
{[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{U,n}"] \arrow[d] & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d] & R\Gamma_{fg}(U,\mathbb{Z}(n)) \arrow[r, dashed] & {[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[d] \\
{[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{X,n}"] \arrow[d] & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, dashed] & {[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[d] \\
{[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{Z,n}"] \arrow[d] & R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d] & R\Gamma_{fg}(Z,\mathbb{Z}(n)) \arrow[r, dashed] & {[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[d] \\
{[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[r] & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n))[1] \arrow[r] & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \arrow[r, dashed] & {[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}]}
\end{tikzcd}$$

For the above diagram, by the axiom (TR3), there are morphisms

$$\begin{aligned}
R\Gamma_{fg}(U,\mathbb{Z}(n)) &\to R\Gamma_{fg}(X,\mathbb{Z}(n)),\\
R\Gamma_{fg}(X,\mathbb{Z}(n)) &\to R\Gamma_{fg}(Z,\mathbb{Z}(n)),\\
R\Gamma_{fg}(Z,\mathbb{Z}(n)) &\to R\Gamma_{fg}(U,\mathbb{Z}(n))[1]
\end{aligned}$$

making everything commute. According to 0.3.6, these arrows are uniquely defined.

(1.8.7)

$$\begin{tikzcd}
{[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{U,n}"] \arrow[d] & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "(a)"] & R\Gamma_{fg}(U,\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d, dashed, "\exists!"] & {[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[d] \\
{[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{X,n}"] \arrow[d] & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "(b)"] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d, dashed, "\exists!"] & {[R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[d] \\
{[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-2]]} \arrow[r, "\alpha_{Z,n}"] \arrow[d] & R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "(c)"] & R\Gamma_{fg}(Z,\mathbb{Z}(n)) \arrow[r, dashed] \arrow[d, dashed, "\exists!"] & {[R\Gamma(Z_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[d] \\
{[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}[-1]]} \arrow[r] & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n))[1] \arrow[r] & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \arrow[r, dashed] & {[R\Gamma(U_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}]}
\end{tikzcd}$$

Obtained this way, the third column

(1.8.8)

$$R\Gamma_{fg}(U,\mathbb{Z}(n)) \to R\Gamma_{fg}(X,\mathbb{Z}(n)) \to R\Gamma_{fg}(Z,\mathbb{Z}(n)) \to R\Gamma_{fg}(U,\mathbb{Z}(n))[1]$$

is uniquely defined, but a priori it is not a distinguished triangle.

1.8.4. At least in the case $X(\mathbb{R}) = \emptyset$, as we already observed in 1.5.2,

$$\begin{aligned}
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Q}/\mathbb{Z}[-2]) &\simeq R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)),\\
R\Gamma_{fg}(X,\mathbb{Z}(n)) &\simeq R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{Z}[-1]),
\end{aligned}$$

and one easily sees that this actually gives us an isomorphism between (1.8.8) and the distinguished triangle

$$\begin{aligned} &R\mathrm{Hom}(R\Gamma(U_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \to R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \\ &\qquad \to R\mathrm{Hom}(R\Gamma(Z_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}[-1]) \to R\mathrm{Hom}(R\Gamma(U_{ét}, \mathbb{Z}^c(n)), \mathbb{Z}). \end{aligned}$$

In particular, (1.8.8) is distinguished.

1.8.5. In general, as we noted in 1.5.6, tensoring the diagram with $\mathbb{Q}$ and $\mathbb{Z}/m\mathbb{Z}$, gives us isomorphisms

$$\begin{tikzcd}
R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q} \arrow[r, "\cong"] \arrow[d] & R\mathrm{Hom}(R\Gamma(U_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \arrow[d] \\
R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \arrow[r, "\cong"] \arrow[d] & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \arrow[d] \\
R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes \mathbb{Q} \arrow[r, "\cong"] \arrow[d] & R\mathrm{Hom}(R\Gamma(Z_{ét}, \mathbb{Z}^c(n)), \mathbb{Q}[-1]) \arrow[d] \\
R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q}[1] \arrow[r, "\cong"] & R\mathrm{Hom}(R\Gamma(U_{ét}, \mathbb{Z}^c(n)), \mathbb{Q})
\end{tikzcd}$$

and

$$\begin{tikzcd}
R\Gamma_c(U_{ét}, \mathbb{Z}/m(n)) \arrow[r, "\cong"] \arrow[d] & R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \arrow[d] \\
R\Gamma_c(X_{ét}, \mathbb{Z}/m(n)) \arrow[r, "\cong"] \arrow[d] & R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \arrow[d] \\
R\Gamma_c(Z_{ét}, \mathbb{Z}/m(n)) \arrow[r, "\cong"] \arrow[d] & R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \arrow[d] \\
R\Gamma_c(U_{ét}, \mathbb{Z}/m(n))[1] \arrow[r, "\cong"] & R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m[1]
\end{tikzcd}$$

This means that the triangles

$$\begin{aligned} (1.8.9) \quad &R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q} \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes \mathbb{Q} \\ &\qquad \to R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes \mathbb{Q} \to R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes \mathbb{Q}[1] \end{aligned}$$

and

$$\begin{aligned} (1.8.10) \quad &R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \\ &\qquad \to R\Gamma_{fg}(Z, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m \to R\Gamma_{fg}(U, \mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/m[1] \end{aligned}$$

are distinguished.

1.8.6. Let us make use of (1.8.10). For each prime p we may consider the "derived p-adic completions"

$$R\Gamma_{fg}(-,\mathbb{Z}(n))^{\wedge}_p := R\varprojlim_k (R\Gamma_{fg}(-,\mathbb{Z}(n)) \otimes^{\mathbf{L}} \mathbb{Z}/p^k\mathbb{Z}),$$

as discussed in [BS2013] and [Stacks, Tag 091N]. This will give us a distinguished triangle

$$R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_p \to R\Gamma_{fg}(X,\mathbb{Z}(n))^{\wedge}_p \to R\Gamma_{fg}(Z,\mathbb{Z}(n))^{\wedge}_p \to R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_p[1]$$

It induces a long exact sequence in cohomology, which thanks to [Stacks, 0A06] and flatness of $\mathbb{Z}_p$ may be identified with

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^i(R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_p) \arrow[r, "\cong"] \arrow[d] & H^i_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \arrow[d] \\
H^i(R\Gamma_{fg}(X,\mathbb{Z}(n))^{\wedge}_p) \arrow[r, "\cong"] \arrow[d] & H^i_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \arrow[d] \\
H^i(R\Gamma_{fg}(Z,\mathbb{Z}(n))^{\wedge}_p) \arrow[r, "\cong"] \arrow[d] & H^i_{fg}(Z,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \arrow[d] \\
H^{i+1}(R\Gamma_{fg}(U,\mathbb{Z}(n))^{\wedge}_p) \arrow[r, "\cong"] \arrow[d] & H^{i+1}_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Now the exactness of

$$\begin{aligned}\cdots \to H^i_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to H^i_{fg}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to H^i_{fg}(Z,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \\ \to H^{i+1}_{fg}(U,\mathbb{Z}(n)) \otimes \mathbb{Z}_p \to \cdots\end{aligned}$$

for each prime p implies that the sequence

$$\cdots \to H^i_{fg}(U,\mathbb{Z}(n)) \to H^i_{fg}(X,\mathbb{Z}(n)) \to H^i_{fg}(Z,\mathbb{Z}(n)) \to H^{i+1}_{fg}(U,\mathbb{Z}(n)) \to \cdots$$

is exact as well. This uses the fact that the groups $H^i_{fg}(-,\mathbb{Z}(n))$ are finitely generated and $\mathbb{Z}_p$ is flat.

Indeed, given a morphism of finitely generated abelian groups $f\colon A \to B$, one sees that f is an isomorphism if and only if $f \otimes \mathbb{Z}_p\colon A \otimes \mathbb{Z}_p \to B \otimes \mathbb{Z}_p$ is an isomorphism for all p. Now a complex

$$(C^\bullet, f^\bullet)\colon \qquad \cdots \to C^{i-1} \xrightarrow{f^{i-1}} C^i \xrightarrow{f^i} C^{i+1} \to \cdots$$

is acyclic if and only if for each i in the diagram

$$\begin{tikzcd}
& \operatorname{im} f^{i-1} \arrow[rr, tail, dashed, "\exists!"] \arrow[dr, tail] & & \ker f^i \arrow[dl, tail] & \\
C^{i-1} \arrow[ur, two heads] \arrow[rr, "f^{i-1}"] & & C^i \arrow[rr, "f^i"] \arrow[dl, two heads] \arrow[dr, two heads] & & C^{i+1} \\
& \operatorname{coker} f^{i-1} \arrow[rr, two heads, dashed, "\exists!"] & & \operatorname{im} f^i \arrow[ur, tail] &
\end{tikzcd}$$

$\operatorname{im} f^{i-1} \rightarrowtail \ker f^i$ is an isomorphism. Therefore, by the above and flatness of $\mathbb{Z}_p$, if $C^\bullet$ are finitely generated groups, $(C^\bullet, f^\bullet)$ is acyclic if and only if $(C^\bullet \otimes \mathbb{Z}_p, f^\bullet \otimes \mathbb{Z}_p)$ is acyclic for all p.

I suspect that the triangle (1.8.8) is actually distinguished, but the above argument at least settles that (1.8.8) induces a long exact sequence in cohomology, which will be enough for our purposes.

1.8.7. Remark. The argument from 1.8.6 may seem a bit too twisted, but there is a reason for that. We have to apply first $- \otimes^{\mathbf{L}} \mathbb{Z}/p^k\mathbb{Z}$, and then take the limit $R \varprojlim_k$ because

$$R\mathrm{Hom}(R\Gamma((-)_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \otimes^{\mathbf{L}} \mathbb{Z}/m\mathbb{Z} \simeq 0,$$

while the complex

$$R\mathrm{Hom}(R\Gamma((-)_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}[-2]) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq R\mathrm{Hom}(R\Gamma((-)_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{Q}_p[-2])$$

is not trivial. Intuitively, the whole argument comes from faithful flatness of $\widehat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$. We still looked at each p separately to make use of the derived completion $R \varprojlim_k (- \otimes^{\mathbf{L}} \mathbb{Z}/p^k\mathbb{Z})$, which behaves nicely.

$R\Gamma_{W,c}(X, \mathbb{Z}(n))$ and open-closed decompositions

Recall now that the Weil-étale complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ was defined only up to a non-unique isomorphism in the derived category by the distinguished triangle

$$\begin{aligned} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \to R\Gamma_{fg}(X, \mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\ \to R\Gamma_{W,c}(X, \mathbb{Z}(n))[1] \end{aligned}$$

where the morphism i_∞^* is uniquely defined by the commutative triangle

$$\begin{tikzcd}
& R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \arrow[dl] \arrow[dr, "u_\infty^*"] & \\
R\Gamma_{fg}(X, \mathbb{Z}(n)) \arrow[rr, dashed, "i_\infty^*", "\exists !"'] & & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})
\end{tikzcd}$$

(see 1.6.1).

1.8.8. Lemma. *For an open-closed decomposition $U \hookrightarrow X \leftarrow Z$ the morphism u_∞^* gives a morphism between the corresponding distinguished triangles of cohomology with compact support:*

$$\begin{tikzcd}
R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r, "u_{\infty,U}^*"] \arrow[d] \arrow[dr, phantom, "(d)"] & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \arrow[d] \\
R\Gamma_c(X_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r, "u_{\infty,X}^*"] \arrow[d] \arrow[dr, phantom, "(e)"] & R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \arrow[d] \\
R\Gamma_c(Z_{\acute{e}t}, \mathbb{Z}(n)) \arrow[r, "u_{\infty,Z}^*"] \arrow[d] \arrow[dr, phantom, "(f)"] & R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{Z}) \arrow[d] \\
R\Gamma_c(U_{\acute{e}t}, \mathbb{Z}(n))[1] \arrow[r, "u_{\infty,U}^*[1]"] & R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})[1]
\end{tikzcd} \tag{1.8.11}$$

Proof. Follows from the definition of u_∞^* and the fact that α^* is compatible with the triangles associated to open-closed decompositions, as we verified in 0.8.4.

∎

We now may assemble everything into the commutative prism displayed on the next page.

$$
\begin{tikzcd}
 & & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n)) \arrow[dl] \arrow[dd] \arrow[dr, "u^*_{\infty,U}"] & & \\
R\Gamma_{W,c}(U,\mathbb{Z}(n)) \arrow[r, dashed] & R\Gamma_{fg}(U,\mathbb{Z}(n)) \arrow[rr, dashed, "i^*_{\infty,U}", "\exists!"'] \arrow[dd] \arrow[dr, phantom, "(a)"] & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[r, dashed] \arrow[dd] \arrow[dl, phantom, "(d)"] & R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] \\
 & & R\Gamma_c(X_{\acute{e}t},\mathbb{Z}(n)) \arrow[dl] \arrow[dd] \arrow[dr, "u^*_{\infty,X}"] & & \\
R\Gamma_{W,c}(X,\mathbb{Z}(n)) \arrow[r, dashed] & R\Gamma_{fg}(X,\mathbb{Z}(n)) \arrow[rr, dashed, "i^*_{\infty,X}", "\exists!"'] \arrow[dd] \arrow[dr, phantom, "(b)"] & & R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[r, dashed] \arrow[dd] \arrow[dl, phantom, "(e)"] & R\Gamma_{W,c}(X,\mathbb{Z}(n))[1] \\
 & & R\Gamma_c(Z_{\acute{e}t},\mathbb{Z}(n)) \arrow[dl] \arrow[dd] \arrow[dr, "u^*_{\infty,Z}"] & & \\
R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \arrow[r, dashed] & R\Gamma_{fg}(Z,\mathbb{Z}(n)) \arrow[rr, dashed, "i^*_{\infty,Z}", "\exists!"'] \arrow[dd] \arrow[dr, phantom, "(c)"] & & R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \arrow[r, dashed] \arrow[dd] \arrow[dl, phantom, "(f)"] & R\Gamma_{W,c}(Z,\mathbb{Z}(n))[1] \\
 & & R\Gamma_c(U_{\acute{e}t},\mathbb{Z}(n))[1] \arrow[dl] \arrow[dr, "{u^*_{\infty,U}[1]}"] & & \\
R\Gamma_{W,c}(U,\mathbb{Z}(n))[1] \arrow[r, dashed] & R\Gamma_{fg}(U,\mathbb{Z}(n))[1] \arrow[rr, dashed, "{i^*_{\infty,U}[1]}", "\exists!"'] & & R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})[1] \arrow[r, dashed] & R\Gamma_{W,c}(U,\mathbb{Z}(n))[2]
\end{tikzcd}
\tag{1.8.12}
$$

Here the squares (a), (b), (c) are the ones that appear in the diagram (1.8.7); (d), (e), (f) are the ones that appear in (1.8.11); the arrows $i_{\infty,U}^*, i_{\infty,X}^*, i_{\infty,Z}^*$ are the unique morphisms in the derived category that make the triangles commute.

Morally, we should have a distinguished triangle of perfect complexes

$$R\Gamma_{W,c}(U,\mathbb{Z}(n)) \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \to R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \to R\Gamma_{W,c}(U,\mathbb{Z}(n))[1]$$

which would give us then a canonical isomorphism of determinants (1.8.1). However, the diagram (1.8.12) a priori does not give a distinguished triangle for Weil-étale cohomology, it gives only a sequence of morphisms that is not necessarily distinguished (again, recall the discussion in 1.8.2). Let us instead give an ad hoc workaround on the level of determinants.

1.8.9. First let us assume for simplicity that $X(\mathbb{R}) = \emptyset$. Then the complexes

$$R\Gamma_{fg}(U,\mathbb{Z}(n)),\ R\Gamma_{fg}(X,\mathbb{Z}(n)),\ R\Gamma_{fg}(Z,\mathbb{Z}(n)),$$
$$R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}),\ R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}),\ R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})$$

are perfect (they do not have 2-torsion in arbitrarily high degrees), and it makes sense to talk about their determinants. From the corresponding columns in (1.8.12) we obtain canonical isomorphisms

$$\det\nolimits_\mathbb{Z} R\Gamma_{fg}(X,\mathbb{Z}(n)) \cong \det\nolimits_\mathbb{Z} R\Gamma_{fg}(U,\mathbb{Z}(n)) \otimes_\mathbb{Z} \det\nolimits_\mathbb{Z} R\Gamma_{fg}(Z,\mathbb{Z}(n)),$$

$$\begin{aligned}&\det\nolimits_\mathbb{Z} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \cong \\ &\qquad \det\nolimits_\mathbb{Z} R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}) \otimes_\mathbb{Z} \det\nolimits_\mathbb{Z} R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z});\end{aligned}$$

and the rows give us isomorphisms

$$\begin{aligned}&\det\nolimits_\mathbb{Z} R\Gamma_{W,c}(U,\mathbb{Z}(n)) \cong \\ &\qquad \det\nolimits_\mathbb{Z} R\Gamma_{fg}(U,\mathbb{Z}(n)) \otimes_\mathbb{Z} (\det\nolimits_\mathbb{Z} R\Gamma_c(G_\mathbb{R},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))^{-1},\end{aligned}$$

$$\begin{aligned}&\det\nolimits_\mathbb{Z} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \\ &\qquad \det\nolimits_\mathbb{Z} R\Gamma_{fg}(X,\mathbb{Z}(n)) \otimes_\mathbb{Z} (\det\nolimits_\mathbb{Z} R\Gamma_c(G_\mathbb{R},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))^{-1},\end{aligned}$$

$$\begin{aligned}&\det\nolimits_\mathbb{Z} R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \cong \\ &\qquad \det\nolimits_\mathbb{Z} R\Gamma_{fg}(Z,\mathbb{Z}(n)) \otimes_\mathbb{Z} (\det\nolimits_\mathbb{Z} R\Gamma_c(G_\mathbb{R},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}))^{-1}.\end{aligned}$$

Combining all the above, we obtain the desired isomorphism (1.8.1).

1.8.10. Now let us treat the general case, when possibly $X(\mathbb{R}) \neq \emptyset$. The above argument does not quite make sense, because the involved complexes are not bounded above. We consider the morphism of long exact sequences in cohomology given by $H^\bullet(i_\infty^*)$. We know that $H^i(i_\infty^*)$ is an isomorphism for $i \gg 0$, so truncating the long exact sequences at a sufficiently large degree $m \gg 0$, we obtain

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^i_{fg}(U, \mathbb{Z}(n)) & \longrightarrow & H^i_c(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^i_{fg}(X, \mathbb{Z}(n)) & \longrightarrow & H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^i_{fg}(Z, \mathbb{Z}(n)) & \longrightarrow & H^i_c(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow{\scriptstyle \delta^i_{fg}} & & \downarrow{\scriptstyle \delta^i_c} \\
H^{i+1}_{fg}(U, \mathbb{Z}(n)) & \longrightarrow & H^{i+1}_c(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H^m_{fg}(U, \mathbb{Z}(n)) & \xrightarrow{\cong} & H^m_c(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^m_{fg}(X, \mathbb{Z}(n)) & \xrightarrow{\cong} & H^m_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^m_{fg}(Z, \mathbb{Z}(n)) & \xrightarrow{\cong} & H^m_c(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\, \mathbb{Z}) \\
\downarrow{\scriptstyle \delta^m_{fg}} & & \downarrow{\scriptstyle \delta^m_c} \\
\operatorname{coker} \delta^m_{fg} & \xrightarrow{\cong} & \operatorname{coker} \delta^m_c \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\tag{1.8.13}
$$

Note that the horizontal arrows that are isomorphisms induce canonical isomorphisms between the determinants. In particular, there is a canonical isomorphism

$$\det\nolimits_\mathbb{Z} \operatorname{coker} \delta^m_{fg} \cong \det\nolimits_\mathbb{Z} \operatorname{coker} \delta^m_c,$$

and hence a *canonical* isomorphism

$$\det\nolimits_\mathbb{Z} \operatorname{coker} \delta^m_{fg} \otimes_\mathbb{Z} (\det\nolimits_\mathbb{Z} \operatorname{coker} \delta^m_c)^{-1} \cong \mathbb{Z}. \tag{1.8.14}$$

Now the exact columns of (1.8.13) give us canonical isomorphisms

$$(1.8.15)\quad \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n))^{(-1)^i} \cong$$
$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(U,\mathbb{Z}(n))^{(-1)^i} \otimes_{\mathbb{Z}} \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(Z,\mathbb{Z}(n))^{(-1)^i} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \operatorname{coker} \delta^m_{fg}$$

and

$$(1.8.16)\quad \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})^{(-1)^i} \cong$$
$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})^{(-1)^i} \otimes_{\mathbb{Z}}$$
$$\bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})^{(-1)^i} \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \operatorname{coker} \delta^m_c.$$

For fixed distinguished rows as in (1.8.12), we have canonical isomorphisms

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \cong \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{W,c}(X,\mathbb{Z}(n))^{(-1)^i}$$
$$\cong \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_{fg}(X,\mathbb{Z}(n))^{(-1)^i} \otimes_{\mathbb{Z}} \bigotimes_{i\le m} \det_{\mathbb{Z}} H^i_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{Z})^{(-1)^{i+1}},$$

and similarly for U and Z in place of X. Combining these with (1.8.15), (1.8.16), and (1.8.14), gives us a canonical isomorphism (1.8.1).

1.8.11. Remark. A cheap way to get around the above technical problems is to consider the Weil-étale cohomology with coefficients in $\mathbb{Z}[1/2]$, i.e. tensor the distinguished triangle defining $R\Gamma_{W,c}(X,\mathbb{Z}(n))$ with the flat $\mathbb{Z}$-module $\mathbb{Z}[1/2]$. Then the resulting distinguished triangle

$$R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \to R\Gamma_{fg}(U,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$$
$$\to R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{Z}[1/2]) \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2][1]$$

consists of perfect complexes, as we just killed the 2-torsion. But then in the next chapter, we would be able to state the special value conjecture only up to some power of 2.

Chapter 2

Conjecture about zeta-values

The regulator morphism is introduced in §2.2, using the constructions from [KLMS2006]. It is more naturally defined with its target in Deligne homology, and all the necessary preliminaries about it are included in §2.1. Then in everything is put together to formulate the conjectural relation of Weil-étale complexes $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ to the special values $\zeta_X^*(n)$. Finally, it is verified in §2.4 that the conjecture is compatible with disjoint unions, open-closed decompositions and taking the affine bundle $\mathbb{A}_X^r \to X$.

2.1 Deligne cohomology and homology

Now we are going review the definitions of Deligne cohomology and homology. These were introduced in Beĭlinson's seminal paper [Bei1984], so they are also known in the literature as "Deligne–Beĭlinson (co)homology", but I will use the term "Deligne (co)homology" for brevity. For the technical details, the reader may consult [EV1988] and [Jan1988].

For this section, $\mathcal{X}$ denotes a smooth complex algebraic variety over $\mathbb{C}$, and $\mathbb{Z} \subset A \subseteq \mathbb{R}$ denotes a subring of the ring of real numbers (eventually we will be interested in $A = \mathbb{Z}$ and $\mathbb{R}$). For a parameter $k \in \mathbb{Z}$ one can define (co)homology groups

$$H^i_{\mathcal{D}}(\mathcal{X}, A(k)), \quad H_i^{\mathcal{D}}(\mathcal{X}, A(k)).$$

Here k is a "twist" that may be any integer. In fact, for certain values of k the above groups have simpler description, and it will be our case.

We are going to assume that $\mathcal{X}$ is connected, of dimension $d_{\mathbb{C}}$. A **good**

compactification of X is given by

$$X \xhookrightarrow{j} \overline{X} \hookleftarrow D \tag{2.1.1}$$

where $j\colon X \hookrightarrow \overline{X}$ is an embedding into a proper smooth algebraic variety $\overline{X}$, and the complement $D := \overline{X} \setminus X$ is a normal crossing divisor (meaning that locally in the analytic topology, D has smooth components intersecting transversally). Such a good compactification always exists (this follows from Hironaka's resolution of singularities), and we fix one.

Deligne cohomology

We denote by $\Omega^\bullet_{X(\mathbb{C})}$ the de Rham complex of holomorphic differential forms on $X(\mathbb{C})$:

$$0 \to \mathcal{O}_{X(\mathbb{C})} \to \Omega^1_{X(\mathbb{C})} \to \Omega^2_{X(\mathbb{C})} \to \cdots \to \Omega^{d_\mathbb{C}}_{X(\mathbb{C})} \to 0$$

Further, let $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D)$ be the de Rham complex of meromorphic differential forms on $\overline{X}(\mathbb{C})$, holomorphic on $X(\mathbb{C})$, with at most logarithmic poles along $D(\mathbb{C})$. We consider the descending filtration of $\Omega^\bullet_{\overline{X}(\mathbb{C})}(\log D)$ by subcomplexes

$$\begin{aligned}\Omega^{\geqslant k}_{\overline{X}(\mathbb{C})}(\log D)\colon \quad 0 \to \cdots \to 0 \to \Omega^k_{\overline{X}(\mathbb{C})}(\log D) \to \Omega^{k+1}_{\overline{X}(\mathbb{C})}(\log D) \\ \to \cdots \to \Omega^{d_\mathbb{C}}_{\overline{X}(\mathbb{C})}(\log D) \to 0\end{aligned}$$

Let us fix some conventions related to the cones of complexes. If $u\colon A^\bullet \to B^\bullet$ is a morphism of complexes, the corresponding cone complex is given by

$$\mathrm{Cone}(u) := A^\bullet[1] \oplus B^\bullet,$$

together with the differentials

$$\begin{aligned} d^i\colon A^{i+1} \oplus B^i &\to A^{i+2} \oplus B^{i+1}, \\ (a,b) &\mapsto (-d_A^{i+1}(a), u(a) + d_B^i(b)). \end{aligned}$$

This gives us a short exact sequence of complexes

$$B^\bullet \rightarrowtail \mathrm{Cone}(u) \twoheadrightarrow A^\bullet[1]$$

and the corresponding distinguished triangle in the derived category

$$A^\bullet \to B^\bullet \to \mathrm{Cone}(u) \to A^\bullet[1]$$

2.1.1. Definition. Let A be a subring of $\mathbb{R}$. For $k \in \mathbb{Z}$ we denote

$$A(k) := (2\pi i)^k\, A \subset \mathbb{C}.$$

This is a $G_\mathbb{R}$-module, and we will also denote by $A(k)$ the corresponding ($G_\mathbb{R}$-equivariant) sheaf on $\mathcal{X}(\mathbb{C})$. For a fixed good compactification (2.1.1), the corresponding **Deligne–Beĭlinson complex** is the complex of sheaves on $\overline{\mathcal{X}}(\mathbb{C})$ given by

$$A(k)_{\mathcal{D}\text{-}\mathrm{B},(\overline{\mathcal{X}},\mathcal{X})} := \operatorname{Cone}\left(Rj_* A(k) \oplus \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \xrightarrow{\epsilon-\iota} Rj_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})}\right)[-1],$$

where

$$\epsilon\colon Rj_* A(k) \to Rj_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})}$$

is induced by the canonical morphism of sheaves $A(k) \to \mathcal{O}_{\mathcal{X}(\mathbb{C})}$ and

$$\iota\colon \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \to Rj_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})}$$

is induced by a natural inclusion

$$\Omega^\bullet_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \xrightarrow{\simeq} j_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})} = Rj_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})},$$

which is a quasi-isomorphism of filtered complexes (see [Del1971, §3.1]). The corresponding **Deligne cohomology** groups are given by the hypercohomology of $A(k)_{\mathcal{D}\text{-}\mathrm{B},(\overline{\mathcal{X}},\mathcal{X})}$:

$$H^i_\mathcal{D}(\mathcal{X}, A(k)) := H^i(R\Gamma(\overline{\mathcal{X}}(\mathbb{C}), A(k)_{\mathcal{D}\text{-}\mathrm{B},(\overline{\mathcal{X}},\mathcal{X})})).$$

The distinguished triangle of sheaves on $\overline{\mathcal{X}}(\mathbb{C})$

$$A(k)_{\mathcal{D}\text{-}\mathrm{B},(\overline{\mathcal{X}},\mathcal{X})} \to Rj_* A(k) \oplus \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) \xrightarrow{\epsilon-\iota} Rj_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})} \to A(k)_{\mathcal{D}\text{-}\mathrm{B},(\overline{\mathcal{X}},\mathcal{X})}[1]$$

induces the (hyper)cohomology long exact sequence

(2.1.2)
$$\begin{aligned}\cdots \to H^i_\mathcal{D}(\mathcal{X}, A(k)) \to H^i(\mathcal{X}(\mathbb{C}), A(k)) \oplus F^k H^i_{dR}(\mathcal{X}(\mathbb{C})) \xrightarrow{\epsilon-\iota} H^i_{dR}(\mathcal{X}(\mathbb{C}))\\ \to H^{i+1}_\mathcal{D}(\mathcal{X}, A(k)) \to \cdots\end{aligned}$$

where

$$F^k H^i_{dR}(\mathcal{X}(\mathbb{C})) :=$$
$$\operatorname{im}\left(\mathbb{H}^i(\overline{\mathcal{X}}(\mathbb{C}), \Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D)) \hookrightarrow \mathbb{H}^i(\overline{\mathcal{X}}(\mathbb{C}), \Omega^\bullet_{\overline{\mathcal{X}}(\mathbb{C})}(\log D)) \cong H^i_{dR}(\mathcal{X}(\mathbb{C}))\right)$$

denotes the Hodge filtration on the de Rham cohomology of $\mathcal{X}(\mathbb{C})$ (for details about this, see [Del1971] and [Voi2002, Chapter 8]). Using the above distinguished triangle / long exact sequence, one may show that the groups $H^i_{\mathcal{D}}(\mathcal{X}, A(k))$ in fact do not depend on the choice of a good compactification $j\colon \mathcal{X} \hookrightarrow \overline{\mathcal{X}}$ (see [EV1988, Lemma 2.8]). For this we will write simply "$A(k)_{\mathcal{D}\text{-}\text{Б}}$" instead of "$A(k)_{\mathcal{D}\text{-}\text{Б},(\overline{\mathcal{X}},\mathcal{X})}$" if $\mathcal{X}$ is clear from the context and a specific good compactification does not matter.

Eventually we will be interested in a very special case when Deligne cohomology is particularly easy to describe.

2.1.2. Lemma. *For $k > d_{\mathbb{C}}$ and $A = \mathbb{R}$ we have a quasi-isomorphism of complexes*

$$R\Gamma(\overline{\mathcal{X}}(\mathbb{C}), \mathbb{R}(k)_{\mathcal{D}\text{-}\text{Б}}) \simeq R\Gamma(\mathcal{X}(\mathbb{C}), (2\pi i)^{k-1}\,\mathbb{R})[-1].$$

Proof. We have $\Omega^{\geqslant k}_{\overline{\mathcal{X}}(\mathbb{C})}(\log D) = 0$ for $k > d_{\mathbb{C}}$, so that in this case

$$A(k)_{\mathcal{D}\text{-}\text{Б}} = \operatorname{Cone}(Rj_* A(k) \xrightarrow{\epsilon} Rj_*\Omega^\bullet_{\mathcal{X}(\mathbb{C})})[-1] \cong Rj_* \operatorname{Cone}(A(k) \xrightarrow{\epsilon} \Omega^\bullet_{\mathcal{X}(\mathbb{C})})[-1],$$

and we easily see that the complex of sheaves $\operatorname{Cone}(A(k) \xrightarrow{\epsilon} \Omega^\bullet_{\mathcal{X}(\mathbb{C})})[-1]$ on $\mathcal{X}(\mathbb{C})$ is given by

$$[A(k) \to \Omega^\bullet_{\mathcal{X}(\mathbb{C})}[-1]] := \left[0 \to \underset{0}{A(k)} \to \underset{1}{\mathcal{O}_{\mathcal{X}(\mathbb{C})}} \to \underset{2}{\Omega^1_{\mathcal{X}(\mathbb{C})}} \to \underset{3}{\Omega^2_{\mathcal{X}(\mathbb{C})}} \to \cdots \to \underset{d_{\mathbb{C}}+1}{\Omega^{d_{\mathbb{C}}}_{\mathcal{X}(\mathbb{C})}} \to 0\right]$$

—that is, we have the constant sheaf $A(k) := (2\pi i)^k A$ in degree 0, followed by the whole holomorphic de Rham complex on $\mathcal{X}(\mathbb{C})$, shifted by one position. By the Poincaré lemma, we have a quasi-isomorphism of complexes of sheaves on $\mathcal{X}(\mathbb{C})$

$$\mathbb{C} \xrightarrow{\simeq} \Omega^\bullet_{\mathcal{X}(\mathbb{C})}, \tag{2.1.3}$$

and we also have a short exact sequence of $G_{\mathbb{R}}$-modules

$$(2\pi i)^k\,\mathbb{R} \rightarrowtail \mathbb{C} \twoheadrightarrow (2\pi i)^{k-1}\,\mathbb{R}. \tag{2.1.4}$$

Now (2.1.3) and (2.1.4) give us a quasi-isomorphism of complexes of sheaves on $\mathcal{X}(\mathbb{C})$

$$[\mathbb{R}(k) \to \Omega^\bullet_{\mathcal{X}(\mathbb{C})}[-1]] \simeq (2\pi i)^{k-1}\,\mathbb{R}[-1].$$

Putting all this together, we have

$$\begin{aligned} R\Gamma(\overline{X}(\mathbb{C}), \mathbb{R}(k)_{\mathcal{D}\text{-Б}}) &\simeq R\Gamma(\overline{X}(\mathbb{C}), Rj_*[(2\pi i)^k\,\mathbb{R} \to \Omega^\bullet_{X(\mathbb{C})}[-1]]) \\ &\simeq R\Gamma(\overline{X}(\mathbb{C}), Rj_*(2\pi i)^{k-1}\,\mathbb{R})[-1] \\ &\simeq R\Gamma(X(\mathbb{C}), (2\pi i)^{k-1}\,\mathbb{R})[-1]. \end{aligned}$$

■

Deligne homology

Deligne homology $H^{\mathcal{D}}_\bullet(X, A(k))$ is constructed in such a way that there is an isomorphism with Deligne cohomology

$$H^i_{\mathcal{D}}(X, A(k)) \xrightarrow{\cong} H^{\mathcal{D}}_{2d_{\mathbb{C}}-i}(X, A(d_{\mathbb{C}} - k)).$$

To do this, Jannsen in his article [Jan1988] replaces the singular cohomology $H^\bullet(X(\mathbb{C}), A(k))$ with Borel–Moore homology $H^{BM}_\bullet(X(\mathbb{C}), A(k))$, and de Rham cohomology $H^\bullet_{dR}(X(\mathbb{C}))$ with the corresponding object, which he calls "de Rham homology". It would be probably more correct to call $H^{\mathcal{D}}_\bullet(X, A(k))$ "Deligne–Borel–Moore homology".

We would like to compare homological and cohomological complexes, and for this the following convention will be used. To pass from a homological complex $C_\bullet$ to a cohomological complex $'C^\bullet$, we set

$$'C^i := C_{-i},$$

and the differentials are given by

$$('C^i \xrightarrow{d^i} {'C^{i+1}}) := (C_{-i} \xrightarrow{(-1)^{i+1}\,d} C_{-i-1})$$

(note the alternating signs).

As before, we fix a good compactification (2.1.1). Here are the ingredients for the definition of Deligne homology (we refer to [Jan1988] for details).

1. We consider the quotient complex

$$'C^\bullet(\overline{X}, D, A(k)) := {'C^\bullet}(\overline{X}(\mathbb{C}), A(k))/{'C^\bullet_D}(\overline{X}(\mathbb{C}), A(k)),$$

   where $C_\bullet(\overline{X}(\mathbb{C}), A(k))$ denotes the complex of singular C^∞-chains on $\overline{X}(\mathbb{C})$ with coefficients in $A(k) := (2\pi i)^k\,A$, and $C^D_\bullet(\overline{X}(\mathbb{C}), A(k))$ is the subcomplex of chains with support on $D(\mathbb{C})$. We put $'C^\bullet$ instead of $C_\bullet$ to pass to cohomological complexes.

2. We denote by $\Omega^{p,q}_{X(\mathbb{C})^\infty}$ the sheaf of C^∞-(p,q)-forms on $X(\mathbb{C})$ (sometimes also denoted by $\mathcal{A}^{p,q}_{X(\mathbb{C})}$).

3. We denote by $'\Omega^{p,q}_{X(\mathbb{C})^\infty}$ the sheaf of distributions over $\Omega^{-p,-q}_{X(\mathbb{C})^\infty}$. That is, for an open subset $U \subseteq X$ we have

$$'\Omega^{p,q}_{X(\mathbb{C})^\infty}(U) := \{\text{continuous linear functionals on } \Gamma_c(U, \Omega^{-p,-q}_{X(\mathbb{C})^\infty})\}.$$

4. Both $\Omega^{\bullet,\bullet}_{X(\mathbb{C})^\infty}$ and $'\Omega^{\bullet,\bullet}_{X(\mathbb{C})^\infty}$ naturally form double complexes. We denote by $\Omega^{\bullet}_{X(\mathbb{C})^\infty}$ and $'\Omega^{\bullet}_{X(\mathbb{C})^\infty}$ the total complexes associated to $\Omega^{\bullet,\bullet}_{X(\mathbb{C})^\infty}$ and $'\Omega^{\bullet,\bullet}_{X(\mathbb{C})^\infty}$ respectively:

$$\Omega^n_{X(\mathbb{C})^\infty} := \bigoplus_{p+q=n} \Omega^{p,q}_{X(\mathbb{C})^\infty}, \quad '\Omega^n_{X(\mathbb{C})^\infty} := \bigoplus_{p+q=n} {}'\Omega^{p,q}_{X(\mathbb{C})^\infty}.$$

5. As before, we consider the corresponding logarithmic de Rham complexes and their filtrations:

$$\begin{aligned}
\Omega^{\bullet}_{\overline{X}(\mathbb{C})^\infty}(\log D) &:= \Omega^{\bullet}_{\overline{X}(\mathbb{C})}(\log D) \otimes_{\Omega^{\bullet}_{\overline{X}(\mathbb{C})}} \Omega^{\bullet}_{\overline{X}(\mathbb{C})^\infty}, \\
\Omega^{\geqslant k}_{\overline{X}(\mathbb{C})^\infty}(\log D) &:= \Omega^{\geqslant k}_{\overline{X}(\mathbb{C})}(\log D) \otimes_{\Omega^{\bullet}_{\overline{X}(\mathbb{C})}} \Omega^{\bullet}_{\overline{X}(\mathbb{C})^\infty},
\end{aligned}$$

and similarly for $'\Omega$ instead of Ω.

2.1.3. Definition. In the above setting, for a fixed good compactification (2.1.1), consider the complex of abelian groups

$$\begin{aligned}
&'C^{\bullet}_{\mathcal{D}}(\overline{X}, D, A(k)) := \\
&\operatorname{Cone}\left(\begin{matrix} 'C^{\bullet}(\overline{X}, D, A(k)) \\ \oplus \\ \Gamma(\overline{X}(\mathbb{C}), {}'\Omega^{\geqslant k}_{\overline{X}(\mathbb{C})^\infty}(\log D)) \end{matrix} \xrightarrow{\epsilon-\iota} \Gamma(\overline{X}(\mathbb{C}), {}'\Omega^{\bullet}_{\overline{X}(\mathbb{C})^\infty}(\log D)) \right)[-1],
\end{aligned}$$

where ι is induced by the inclusion $'\Omega^{\geqslant k}_{\overline{X}(\mathbb{C})^\infty}(\log D) \subset {}'\Omega^{\bullet}_{\overline{X}(\mathbb{C})^\infty}(\log D)$, and ϵ is given by the integration over chains (see [Jan1988] for details). The corresponding **Deligne homology** groups are given by

$$'H^i_{\mathcal{D}}(X, A(k)) := H^i('C^{\bullet}_{\mathcal{D}}(\overline{X}, D, A(k))).$$

To understand the above definition, we should examine what each complex computes.

1. According to [Jan1988, Lemma 1.11], the complex ${}'C^\bullet(\overline{X}, D, A(k))$ calculates Borel–Moore homology of $X(\mathbb{C})$ with coefficients in $A(k)$: there are canonical isomorphisms

$$H^i({}'C^\bullet(\overline{X}, D, A(k))) \cong {}'H^i_{BM}(X(\mathbb{C}), A(k)) = H^{BM}_{-i}(X(\mathbb{C}), A(-k))$$

(see loc. cit. and [Ver1976, §1] for details on Borel–Moore homology).

2. According to [Jan1988, Corollary 1.13], there are quasi-isomorphisms of fine sheaves

$$\begin{aligned} Rj_*\,{}'\Omega^\bullet_{X(\mathbb{C})^\infty} = j_*\,{}'\Omega^\bullet_{X(\mathbb{C})^\infty} \xleftarrow{\simeq} j_*\Omega^\bullet_{X(\mathbb{C})^\infty}[2d_{\mathbb{C}}] \xleftarrow{\simeq} \Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D)[2d_{\mathbb{C}}] \\ \xrightarrow{\simeq} {}'\Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D) \end{aligned}$$

and then Jannsen defines

$${}'H^i_{dR}(X(\mathbb{C})) := H^i(\Gamma(X(\mathbb{C}), {}'\Omega^\bullet_{X(\mathbb{C})^\infty})) \cong H^i(\Gamma(\overline{X}(\mathbb{C}), {}'\Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D)))$$

to be the **de Rham homology** of $X(\mathbb{C})$ (this is by no means standard terminology).

3. De Rham homology carries a Hodge filtration defined by

$$\begin{aligned} &F^k\,{}'H^i_{dR}(X(\mathbb{C})) := \\ &\mathrm{im}\Big(H^i(\Gamma(\overline{X}(\mathbb{C}), {}'\Omega^{\geqslant k}_{\overline{X}(\mathbb{C})^\infty}(\log D))) \hookrightarrow H^i(\Gamma(\overline{X}(\mathbb{C}), {}'\Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D))) \\ &\qquad\qquad \cong {}'H^i_{dR}(X(\mathbb{C}))\Big) \end{aligned}$$

(the fact that this map is injective is in a sense dual to the corresponding fact for the Hodge filtration on de Rham cohomology).

The above considerations and the definition of Deligne homology give us the long exact sequence

$$\begin{aligned} \text{(2.1.5)} \quad \cdots \to {}'H^i_{\mathcal{D}}(X, A(k)) \to {}'H^i_{BM}(X(\mathbb{C}), A(k)) \oplus F^k\,{}'H^i_{dR}(X(\mathbb{C})) \\ \xrightarrow{\epsilon - \iota} {}'H^i_{dR}(X(\mathbb{C})) \to {}'H^{i+1}_{\mathcal{D}}(X, A(k)) \to \cdots \end{aligned}$$

from which one may see that the groups ${}'H^i_{\mathcal{D}}(X, A(k))$ do not depend on the choice of a good compactification $X \hookrightarrow \overline{X}$ (again, see [Jan1988, Corollary 1.13]).

Twisted Poincaré duality

According to [Jan1988, Theorem 1.15], Deligne cohomology and homology are related through the "twisted Poincaré duality"*

$$H_{\mathcal{D}}^{2d_{\mathbb{C}}+i}(X, A(d_{\mathbb{C}}+k)) \xrightarrow{\cong} {}'H_{\mathcal{D}}^{i}(X, A(k)). \tag{2.1.6}$$

In fact, Jannsen establishes a quasi-isomorphism of complexes of abelian groups

$$R\Gamma(\overline{X}(\mathbb{C}), A(k+d_{\mathbb{C}})_{\mathcal{D}\text{-B},(\overline{X},X)}[2d_{\mathbb{C}}]) \simeq {}'C_{\mathcal{D}}^{\bullet}(\overline{X}, D, A(k)), \tag{2.1.7}$$

where the left hand side computes $H_{\mathcal{D}}^{2d_{\mathbb{C}}+i}(X, A(d_{\mathbb{C}}+k))$ (definition 2.1.1) and the right hand side computes ${}'H_{\mathcal{D}}^{i}(X, A(k))$ (definition 2.1.3). The duality is best understood if we use the homological numbering

$$H_i^{\mathcal{D}}(X, A(k)) := {}'H_{\mathcal{D}}^{-i}(X, A(-k))$$

(sic! the sign of the twist gets flipped as well), and also look at the isomorphism of the long exact sequences (2.1.2) and (2.1.5) (see [Jan1988, Remark 1.16 b)]). The duality takes the familiar form

$$H_{\mathcal{D}}^{i}(X, A(k)) \xrightarrow{\cong} H^{\mathcal{D}}_{2d_{\mathbb{C}}-i}(X, A(d_{\mathbb{C}}-k)).$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H_{\mathcal{D}}^{i}(X, A(k)) \arrow[r, "\cong"] \arrow[d] & H^{\mathcal{D}}_{2d_{\mathbb{C}}-i}(X, A(d_{\mathbb{C}}-k)) \arrow[d] \\
H^i(X(\mathbb{C}), A(k)) \oplus F^k H^i_{dR}(X(\mathbb{C})) \arrow[r, "\cong"] \arrow[d, "\epsilon-\iota"] & H^{BM}_{2d_{\mathbb{C}}-i}(X(\mathbb{C}), A(d_{\mathbb{C}}-k)) \oplus F_{d_{\mathbb{C}}-k} H^{dR}_{2d_{\mathbb{C}}-i}(X(\mathbb{C})) \arrow[d, "\epsilon-\iota"] \\
H^i_{dR}(X(\mathbb{C})) \arrow[r, "\cong"] \arrow[d] & H^{dR}_{2d_{\mathbb{C}}-i}(X(\mathbb{C})) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

As in 2.1.2, eventually we will be interested in a very special case where the Hodge filtration part does not enter.

*The word "twisted" means that the isomorphism takes into account the twist $k \in \mathbb{Z}$. However, this duality is also twisted in the sense that, unlike the usual Poincaré duality, it does not come from some nondegenerate pairing.

2.1.4. Lemma. *For $k > 0$ and $A = \mathbb{R}$ we have a quasi-isomorphism of complexes*

$$'C^\bullet_{\mathcal{D}}(\overline{X}, D, A(k)) \simeq R\mathrm{Hom}(R\Gamma_c(\mathcal{X}(\mathbb{C}), (2\pi i)^{1-k}\,\mathbb{R}), \mathbb{R})[-1] \\ =: R\Gamma_{BM}(\mathcal{X}(\mathbb{C}), (2\pi i)^{1-k}\,\mathbb{R})[-1].$$

Proof. The right hand side calculates Bore–Moore homology, which is by definition dual to cohomology with compact support. In case $k > 0$ we have $'\Omega^{\ge k}_{\overline{X}(\mathbb{C})^\infty}(\log D) = 0$, and the Deligne homology complex is defined by

$$'C^\bullet_{\mathcal{D}}(\overline{X}, D, \mathbb{R}(k)) \\ := \mathrm{Cone}\left('C^\bullet(\overline{X}, D, (2\pi i)^k\,\mathbb{R}) \xrightarrow{\epsilon} \Gamma(\overline{X}(\mathbb{C}), '\Omega^\bullet_{\overline{X}(\mathbb{C})^\infty}(\log D))\right)[-1]$$

Probably the correct way to obtain the result would be to analyze this directly and argue as in 2.1.2. There the map ϵ was essentially the comparison between singular cohomology and de Rham cohomology of $\mathcal{X}(\mathbb{C})$, and in the present situation there should be a similar comparison between Borel–Moore homology and cohomology of $'\Omega^\bullet$, which is dual to the de Rham cohomology with compact support.

As a shortcut, let us assume that $\mathcal{X}(\mathbb{C})$ is connected of dimension $2d_{\mathbb{C}}$. The quasi-isomorphism (2.1.7) together with the quasi-isomorphism from 2.1.2 and the Poincaré duality (in the correct version that takes into account the twists) give us

$$'C^\bullet_{\mathcal{D}}(\overline{X}, D, \mathbb{R}(k)) \simeq R\Gamma(\mathcal{X}(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}-(1-k)}\,\mathbb{R})[2d_{\mathbb{C}} - 1] \\ \simeq R\mathrm{Hom}(R\Gamma_c(\mathcal{X}(\mathbb{C}), (2\pi i)^{1-k}\,\mathbb{R}), \mathbb{R})[-1].$$

If X is not connected, the above may be done separately for the connected components. ∎

I still note that the above argument does the trick and uses only the arguments from Jannsen's paper, but it is *morally wrong*: Jannsen derives (2.1.7) from the Poincaré duality, and in the above proof we applied the duality again.

2.2 The regulator morphism

Now as always in this text, X denotes a scheme over $\operatorname{Spec}\mathbb{Z}$; separated of finite type. At this point we also assume that $X_{\mathbb{C}}$ is smooth, quasi-projective. Let us also assume for the moment that X is of pure dimension d, so that

$$d_{\mathbb{C}} := \dim_{\mathbb{C}} X_{\mathbb{C}} = d - 1.$$

However, later on we will see that this assumption is superficial. We fix a good compactification

$$X_{\mathbb{C}} \xhookrightarrow{j} \overline{X_{\mathbb{C}}} \hookleftarrow D$$

The regulators for higher Chow groups $CH^n(X, p) := H^{2n-p}(X_{ét}, \mathbb{Z}(n))$ were introduced by Bloch in [Blo1986b] as morphisms

$$H^\bullet(X_{ét}, \mathbb{Z}(n)) \to H^\bullet(X_{\mathbb{C}}, \mathbb{Z}(n)) \to H^\bullet_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(n)).$$

Here we are going to use the construction from [KLMS2006] which is given on the level of complexes, not merely separate cohomology groups. The reader is advised to review §0.11 for the definitions of different cycle complexes $z_r(-, -\bullet)$, $z^r(-, -\bullet)$, $z^r_\square(-, -\bullet)$, which will all be used now.

The construction from [KLMS2006, §5.9] gives us a morphism of complexes

$$z^r_{\square,\mathbb{R}}(\overline{X_{\mathbb{C}}}, -\bullet)/z^{r-1}_{\square,\mathbb{R}}(D, -\bullet) \to {}'C^{2r-2d_{\mathbb{C}}+\bullet}_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, X, \mathbb{Z}(r - d_{\mathbb{C}})).$$

Here $z^r_{\square,\mathbb{R}}(-, -\bullet)$ are certain subcomplexes of the cubical cycle complexes $z^r_\square(-, -\bullet)$; I refer to [KLMS2006, §5.4] for the precise definition. According to [KLMS2006, §5.9], there are quasi-isomorphisms

$$z^r_{\square,\mathbb{R}}(\overline{X_{\mathbb{C}}}, -\bullet)/z^{r-1}_{\square,\mathbb{R}}(D, -\bullet) \xrightarrow{\simeq} z^r_\square(\overline{X_{\mathbb{C}}}, -\bullet)/z^{r-1}_\square(D, -\bullet) \xrightarrow{\simeq} z^r_\square(X_{\mathbb{C}}, -\bullet),$$

and finally, we have an isomorphism in the derived category

$$z^r_\square(X_{\mathbb{C}}, -\bullet) \cong z^r(X_{\mathbb{C}}, -\bullet).$$

All this means that in the derived category, we may treat the morphism of Kerr, Lewis, and Müller-Stach as

$$z^r(X_{\mathbb{C}}, -\bullet) \to {}'C^{2r-2d_{\mathbb{C}}+\bullet}_{\mathcal{D}}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(r - d_{\mathbb{C}})). \tag{2.2.1}$$

It gives a "regulator" in the following sense. Taking the corresponding $(-i)$-th cohomology groups and using the duality (2.1.6), we obtain

$$AJ\colon CH^r(X_{\mathbb{C}}, i) \to {}'H^{2r-2d_{\mathbb{C}}-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(r - d_{\mathbb{C}})) \xleftarrow{\cong} H^{2r-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(r)).$$

According to [KLMS2006, §5.5], if $X_{\mathbb{C}}$ is projective, then the composition

$$CH^r(X_{\mathbb{C}}, i) \xrightarrow{AJ} H^{2r-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(r)) \xrightarrow{\pi_{\mathbb{R}}} H^{2r-i}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r))$$

coincides with the regulator defined by Goncharov in [Gon1995].

We consider (2.2.1) for $r = d - n$, where d is the dimension of X and $n < 0$ as always denotes a strictly negative integer. We obtain

$$z^{d-n}(X_{\mathbb{C}}, -\bullet) \to {}'C_{\mathcal{D}}^{2-2n+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n)),$$

which we may also write as

$$R\Gamma(X_{\mathbb{C},Zar}, \mathcal{Z}_{X_{\mathbb{C}}}^{d-n}[2n]) \cong z^{d-n}(X_{\mathbb{C}}, -\bullet)[2n] \to {}'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n))$$

(the first isomorphism is 0.11.9). We consider now the composition

$$\begin{aligned} R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) = R\Gamma(X_{\acute{e}t}, \mathcal{Z}_X^{d-n}[2n]) &\to R\Gamma(X_{Zar}, \mathcal{Z}_X^{d-n}[2n]) \\ \to R\Gamma(X_{\mathbb{C},Zar}, \mathcal{Z}_{X_{\mathbb{C}}}^{d-n}[2n]) &\to {}'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n)) \\ &\xrightarrow{\pi_{\mathbb{R}}} {}'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{R}(1-n)) \end{aligned}$$

As $n < 0$, the target complex may be simplified thanks to 2.1.4:

$${}'C_{\mathcal{D}}^{2+\bullet}(\overline{X_{\mathbb{C}}}, D, \mathbb{Z}(1-n)) \simeq R\Gamma_{BM}(\mathcal{X}(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1]$$

Taking $G_{\mathbb{R}}$-invariants (all the complexes involved in the definitions of Deligne (co)homology and all statements about them are $G_{\mathbb{R}}$-equivariant) we obtain a morphism

(2.2.2) $$Reg\colon R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1].$$

2.2.1. Remark. This suggests that in our situation $n < 0$ the regulator probably has en easier definition which could work under weaker assumptions on $X_{\mathbb{C}}$.

In what follows, we are going to use the $\mathbb{R}$-dual to (2.2.2):

(2.2.3) $$Reg^{\vee}\colon R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}).$$

Compatibility of the regulator with basic operations on schemes

2.2.2. Lemma (Compatibility of the regulator with open-closed decompositions). *Suppose that we have an open-closed decomposition of arithmetic schemes $U \hookrightarrow X \leftarrow Z$ such that $U_{\mathbb{C}}$, $X_{\mathbb{C}}$, $Z_{\mathbb{C}}$ are smooth, quasi-projective varieties. Then* *the corresponding regulator morphisms yield a morphism of distinguished triangles*

$$\begin{tikzcd}
R\Gamma(Z_{ét}, \mathbb{Z}^c(n)) \arrow[r, "Reg_Z"] \arrow[d] & R\Gamma_{BM}(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1] \arrow[d] \\
R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \arrow[r, "Reg_X"] \arrow[d] & R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1] \arrow[d] \\
R\Gamma(U_{ét}, \mathbb{Z}^c(n)) \arrow[r, "Reg_U"] \arrow[d] & R\Gamma_{BM}(G_\mathbb{R}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1] \arrow[d] \\
R\Gamma(Z_{ét}, \mathbb{Z}^c(n))[1] \arrow[r, "Reg_Z[1]"] & R\Gamma_{BM}(G_\mathbb{R}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[2]
\end{tikzcd}$$

Proof. This follows from the functoriality of the construction of Kerr, Lewis, and Müller-Stach with respect to proper and flat morphisms, as discussed in [Wei2017, §3]. ∎

2.2.3. Lemma (Compatibility of the regulator with affine bundles). *For an arithmetic scheme X such that $X_\mathbb{C}$ is smooth and quasi-projective, consider the affine space of dimension r over X and the corresponding set of complex points:*

$$\begin{tikzcd}
\mathbb{A}^r_X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^r \arrow[d] \\
X \arrow[r] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}^r_X(\mathbb{C}) \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^r(\mathbb{C}) \arrow[d] \\
X(\mathbb{C}) \arrow[r] & *
\end{tikzcd}$$

There is a commutative diagram

$$\begin{tikzcd}
R\Gamma_c(G_\mathbb{R}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \arrow[r, "\cong"] \arrow[d, "Reg^\vee_{\mathbb{A}^r_X,n}"'] & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})[-2r-1] \arrow[d, "Reg^\vee_{X,n-r}[-2r]"] \\
R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n)), \mathbb{R}) \arrow[r, "\cong"] & R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n-r)), \mathbb{R})[-2r]
\end{tikzcd}$$

Proof. The diagram is the $\mathbb{R}$-dual to

$$\begin{tikzcd}
R\Gamma_{BM}(G_\mathbb{R}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{Z})[1] & R\Gamma_{BM}(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{Z})[2r+1] \arrow[l, "\cong"'] \\
R\Gamma(\mathbb{A}^r_{X,ét}, \mathbb{Z}^c(n)) \arrow[u, "Reg_{\mathbb{A}^r_X,n}"] & R\Gamma(X_{ét}, \mathbb{Z}^c(n-r))[2r] \arrow[l, "\cong"'] \arrow[u, "Reg_{X,n-r}[2r]"']
\end{tikzcd}$$

so it will be enough to check that the latter tensored with $\mathbb{R}$ commutes, which amounts to the commutativity of the following diagrams of $\mathbb{R}$-vector spaces:

$$\begin{tikzcd} H^{\bullet+1}_{BM}(G_{\mathbb{R}}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) & H^{\bullet+2r+1}_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R}) \arrow[l, "\cong"'] \\ H^\bullet(\mathbb{A}^r_X, \mathbb{R}^c(n)) \arrow[u, "Reg_{\mathbb{A}^r_X,n}"] & H^{\bullet+2r}(X, \mathbb{R}^c(n-r)) \arrow[l, "\cong"'] \arrow[u, "Reg_{X,n-r}[2r]"'] \end{tikzcd}$$

Now on the level of separate cohomology groups, we may use Bloch's construction from [Blo1986b]. Namely, after unwinding our definitions, everything amounts to checking that Bloch's regulator is compatible with the "homotopy isomorphisms" for the cycle complex cohomology and Deligne cohomology:

$$\begin{tikzcd} H^\bullet_{\mathcal{D}}(\mathbb{A}^1 \times X_{\mathbb{C}}, \mathbb{R}(n)) & H^\bullet_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(n)) \arrow[l, "\cong"'] \\ H^\bullet(\mathbb{A}^1 \times X_{\mathbb{C}}, \mathbb{R}(n)) \arrow[u, "\text{Bloch's reg.}"] & H^\bullet(X_{\mathbb{C}}, \mathbb{R}(n)) \arrow[l, "\cong"'] \arrow[u, "\text{Bloch's reg.}"'] \end{tikzcd}$$

■

The regulator conjecture

In order to relate the regulator to our machinery, we need make the following assumption.

2.2.4. Conjecture $\mathbf{B}(X, n)$. For an arithmetic scheme X and $n < 0$, the morphism $Reg^\vee$ (the $\mathbb{R}$-dual of the regulator) is an isomorphism in the derived category.

2.2.5. Remark. This is a standard but very strong assumption. For instance, if X is defined over a finite field, then $X(\mathbb{C}) = \emptyset$, and the conjecture implies that the cohomology groups $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are torsion.

2.2.6. Theorem. *Let X be an arithmetic scheme such that $X_{\mathbb{C}}$ is a smooth quasi-projective variety. Let $n < 0$ be a strictly negative integer for which the conjecture $\mathbf{B}(X, n)$ holds. Then there exists a morphism*

$$\smile\theta\colon R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}[1]$$

giving a long exact sequence

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$$
$$\xrightarrow{\smile\theta} H^{i+2}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

i.e. turning $H^\bullet_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$ into an acyclic complex of finite dimensional vector spaces.

Proof. Recall that according to 1.7.1, we have isomorphisms

(2.2.4)
$$R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1].$$

Using this and the morphism $Reg^\vee$, we may define θ in the obvious way:

$$\begin{array}{c}
R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \\
\downarrow{\scriptstyle\cong} \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \ \oplus \ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1] \\
\downarrow \\
R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1] \\
\downarrow{\scriptstyle Reg^\vee} \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \\
\downarrow \\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \ \oplus \ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \\
\downarrow{\scriptstyle\cong} \\
R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}[1]
\end{array}$$

On the level of cohomology, these morphisms give us

$$\smile \theta\colon H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}.$$

If $Reg^\vee$ is a quasi-isomorphism, we obtain an exact sequence

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+2}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots$$

Indeed, let us denote for brevity

$$\begin{aligned}
A^\bullet &:= R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1], \\
B^\bullet &:= R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1].
\end{aligned}$$

Then $Reg^\vee$ conjecturally gives isomorphisms $H^i(B^\bullet) \xrightarrow{\cong} H^{i+1}(A^\bullet)$, and the above sequence looks like

$$\begin{tikzcd}[column sep=small] \cdots & H^i(A^\bullet) \oplus H^i(B^\bullet) \arrow[r, "\cong"] & H^{i+1}(A^\bullet) \oplus H^{i+1}(B^\bullet) \arrow[r, "\cong"] & H^{i+2}(A^\bullet) \oplus H^{i+2}(B^\bullet) & \cdots \end{tikzcd}$$

which is clearly exact. ∎

2.3 The conjecture $\mathbf{C}(X,n)$

In the previous section we built a morphism

$$\smile \theta\colon R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \to R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R}[1]$$

that produces an acyclic complex of finitely generated $\mathbb{R}$-vector spaces

$$H^\bullet_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R}.$$

This means that there is a canonical trivialization isomorphism

$$\begin{aligned} \text{(2.3.1)}\quad \lambda\colon \mathbb{R} \xrightarrow{\cong} \det\nolimits_{\mathbb{R}} H^\bullet_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} &\xrightarrow{\cong} \det\nolimits_{\mathbb{R}} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \\ &\xrightarrow{\cong} (\det\nolimits_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes \mathbb{R}. \end{aligned}$$

Another way to get the same morphism is to go back to the definition of $\smile \theta$ and recall that it uses the splitting

$$\begin{aligned} \text{(2.3.2)}\quad R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1]) \oplus R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \\ \xrightarrow{\cong} R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \end{aligned}$$

and the quasi-isomorphism

$$Reg^\vee\colon R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \xrightarrow{\simeq} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}).$$

These two give us an isomorphism
(2.3.3)

$$\begin{tikzcd} \begin{matrix} R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-2] \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \end{matrix} \arrow[r, "Reg^\vee[-1]\oplus \mathrm{id}", "\cong"'] \arrow[dr, dashed] & \begin{matrix} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R}[-1]) \\ \oplus \\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1] \end{matrix} \arrow[d, "\cong"', "(2.3.2)"] \\ & R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R} \end{tikzcd}$$

which after taking the determinants gives us a canonical isomorphism

(2.3.4)
$$\lambda\colon \mathbb{R} \xrightarrow{\cong} (\det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) \otimes_{\mathbb{R}} (\det_{\mathbb{R}} R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}))^{-1}$$
$$\xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Now in terms of the trivialization morphism λ, we are ready to formulate our main conjecture, which is similar to [Mor2014, Conjecture 4.2] and [FM2016, Conjecture 5.12, 5.13].

2.3.1. Conjecture $\mathbf{C}(X,n)$. For an arithmetic scheme X and $n < 0$

a) assume that the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ holds;

b) assume that $X_{\mathbb{C}}$ is smooth, quasi-projective, so that the regulator morphism $Reg^\vee$ exists; assume that the conjecture $\mathbf{B}(X,n)$ holds;

c) assume that the zeta-function of X
$$\zeta(X,s) := \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}}$$
has a meromorphic continuation near $s = n$.

Then

1) the leading coefficient $\zeta^*(X,n)$ of the Taylor expansion of $\zeta(X,s)$ at $s = n$ is given up to sign by
$$\lambda(\zeta^*(X,n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)),$$
where λ is the trivialization morphism defined in (2.3.1);

2) the vanishing order of $\zeta(X,n)$ at $s = n$ is given by the weighted alternating sum of ranks of $H^i_{W,c}(X, \mathbb{Z}(n))$:

(2.3.5)
$$\operatorname{ord}_{s=n} \zeta(X,s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)).$$

2.3.2. Remark. The sum in (2.3.5) is finite, because as we saw in 1.6.8, the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ implies that the complex $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is perfect.

Since the conjectures $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X,n)$ imply that the groups
$$H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}$$

form an acyclic complex, the usual Euler characteristic of $R\Gamma_{\mathrm{W,c}}(X, \mathbb{Z}(n))$ vanishes:

$$\begin{aligned}\chi(R\Gamma_{\mathrm{W,c}}(X, \mathbb{Z}(n))) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)) \\ = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} = 0.\end{aligned}$$

The sum in (2.3.5) is known as the **secondary Euler characteristic**:

$$\chi'(C^\bullet) := \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk} H^i(C^\bullet).$$

For a distinguished triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

usually

$$\chi'(B^\bullet) \neq \chi'(A^\bullet) + \chi'(C^\bullet),$$

unless the triangle is split, but the secondary Euler characteristic is still a natural invariant for acyclic complexes and it arises in various natural contexts; see [Ram2016].

2.3.3. Remark. The parts 1) and 2) of the conjecture $\mathbf{C}(X, n)$ are equivalent to Conjecture 5.12 and Conjecture 5.13 from [FM2016] if X is proper and regular. This is rather straightforward to see by going through the constructions of Flach and Morin and comparing them to our constructions. Then it is showed in [FM2016, §5.6] that their conjecture 5.12 is compatible with the Tamagawa number conjecture.

2.3.4. Proposition. *Assuming the conjectures $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X, n)$, the weighted sum of ranks of $H^i_{W,c}(X, \mathbb{Z}(n))$ equals the Euler characteristic of*

$$R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R});$$

that is,

$$\begin{aligned}\sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rk}_{\mathbb{Z}} H^i_{W,c}(X, \mathbb{Z}(n)) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \\ =: \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})).\end{aligned}$$

Proof. Thanks to the splitting

$$\begin{aligned}R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R} \cong \\ R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[-1]\end{aligned}$$

and the quasi-isomorphism

$$Reg^\vee \colon R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \xrightarrow{\cong} R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}),$$

we have

$$\begin{aligned} &R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong \\ &\qquad R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \oplus R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-2], \end{aligned}$$

so that

$$H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \oplus H^{i-2}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}).$$

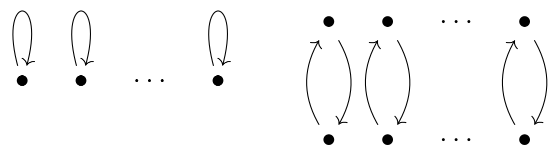
Now

$$\begin{aligned} &\sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \\ &\qquad = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} (H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ &\qquad\qquad + \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} (H^{i-2}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \end{aligned}$$

$$\begin{aligned} &= \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \dim_\mathbb{R} H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ &\qquad - \sum_{i\in\mathbb{Z}} (-1)^i \cdot (i+1) \cdot \dim_\mathbb{R} H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ &\qquad = -\sum_{i\in\mathbb{Z}} (-1)^i \dim_\mathbb{R} H^{i-1}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \\ &\qquad\qquad = \sum_{i\in\mathbb{Z}} (-1)^i \dim_\mathbb{R} H^{i}_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}). \end{aligned}$$

■

2.3.5. Elementary example. Here is one easy illustration for 2.3.4. If $X = \operatorname{Spec} O_K$ is a number ring, then the space $X(\mathbb{C})$ consists of $r_1 + 2\,r_2$ points, corresponding to the real places of K and complex places coming in conjugate pairs:



r_1 points $\qquad$ $2\,r_2$ points

Now $R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})$ in this case may be identified with the complex having just a single $G_\mathbb{R}$-module in degree 0, namely

$$((2\pi i)^n\,\mathbb{R})^{\oplus r_1} \oplus ((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2},$$

where $G_\mathbb{R}$ acts on $((2\pi i)^n\,\mathbb{R})^{\oplus r_1}$ by complex conjugation, while the action on $((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2}$ is given by $(z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$ on each summand $(2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R}$. If n is odd, then the action of $G_\mathbb{R}$ on $((2\pi i)^n\,\mathbb{R})^{\oplus r_1}$ has no fixed points, and if n is even, this action is trivial. As for the other part $((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2}$, we see that the space of $G_\mathbb{R}$-fixed points has real dimension r_2, regardless of the parity of n. Thus in this case

$$H^i_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) = \begin{cases} r_2, & n \text{ odd},\ i = 0; \\ r_1 + r_2, & n \text{ even},\ i = 0; \\ 0 & i \neq 0. \end{cases}$$

Therefore

$$\chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}. \end{cases}$$

This agrees with the vanishing order of the Dedekind zeta function $\zeta(\operatorname{Spec} O_K, s)$ at strictly negative integers.

2.3.6. Trivial example. If X is a variety over $\mathbb{F}_q$, then

$$\zeta(X, s) = Z(X, q^{-s}),$$

where

$$Z(X, t) := \exp\left(\sum_{k\geq 1} \frac{\#X(\mathbb{F}_{q^k})}{k}\, t^k\right)$$

is Weil zeta function. Now if $\zeta(X, s)$ has a zero or pole at s, we have necessarily

$$\operatorname{Re} s = i/2, \quad 0 \leq i \leq 2 \dim X$$

—this may be seen from Weil's conjectures (see e.g. [Kat1994, p. 26–27]). In particular, there are no zeros nor poles for $s < 0$, and the identity (2.3.4) is trivially correct in this case:

$$\operatorname{ord}_{s=n} \zeta(X, s) = 0 = \chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})).$$

2.4 Stability of the conjecture under some operations on schemes

The following properties are clear from the definition of the zeta function of an arithmetic scheme:

1) If $U \hookrightarrow X \hookleftarrow Z$ is an open-closed decomposition, then

$$\zeta(X,s) = \zeta(U,s) \cdot \zeta(Z,s). \tag{2.4.1}$$

2) For $r \geq 0$, consider the affine space $\mathbb{A}^r_X := \mathbb{A}^r_{\mathbb{Z}} \times X$. Then

$$\zeta(\mathbb{A}^r_X, s) = \zeta(X, s-r). \tag{2.4.2}$$

This suggests that our conjecture $\mathbf{C}(X,n)$ should also be compatible with open-closed decompositions and considering the affine space over X. Our goal is to verify that. We need to establish several lemmas.

2.4.1. Lemma. *The morphism λ is compatible with open-closed decompositions $U \hookrightarrow X \hookleftarrow Z$. Such a decomposition gives a commutative diagram*

$$\begin{tikzcd}
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \arrow[r, "x\otimes y \mapsto xy", "\cong"'] \arrow[d, "\lambda_U \otimes \lambda_Z"', "\cong"] & \mathbb{R} \arrow[d, "\cong"', "\lambda_X"] \\
\begin{array}{c}(\det_{\mathbb{Z}} R\Gamma_{W,c}(U,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R} \\ \otimes_{\mathbb{R}} \\ (\det_{\mathbb{Z}} R\Gamma_{W,c}(Z,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}\end{array} \arrow[r, "\cong"] & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))) \otimes_{\mathbb{Z}} \mathbb{R}
\end{tikzcd}$$

Where the bottom row is induced by the canonical isomorphism from 1.8.1.

Proof. This follows from the compatibility of the regulator with open-closed decompositions (see 2.2.2) and the ad-hoc isomorphism

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(U,\mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma_{W,c}(Z,\mathbb{Z}(n)) \cong \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))$$

constructed in 1.8.1. ■

2.4.2. Lemma. *The morphism λ is compatible with affine bundles. We have a commutative diagram*

$$\begin{tikzcd}
& \mathbb{R} \arrow[dl, "\lambda_{\mathbb{A}^r_X}"'] \arrow[dr, "\lambda_X"] & \\
(\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathbb{A}^r_X,\mathbb{Z}(n)) \otimes \mathbb{R} \arrow[rr, "\cong"] & & (\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n-r)) \otimes \mathbb{R}
\end{tikzcd}$$

Proof. Follows from 2.2.3. ■

2.4.3. Lemma. *There is a quasi-isomorphism*

$$R\Gamma_{BM}(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \simeq R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\, \mathbb{R})[2r]$$

or dually,

$$R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \simeq R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\, \mathbb{R})[-2r]. \tag{2.4.3}$$

Proof. We already assumed that $X_{\mathbb{C}}$ is smooth to formulate the conjecture. Further, let us assume for simplicity that $X(\mathbb{C})$ is connected of dimension $d_{\mathbb{C}}$. Then Poincaré duality tells us that

$$\begin{aligned} &R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \simeq \\ &\qquad\qquad R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}+r-n}\, \mathbb{R}), \mathbb{R}[-2d_{\mathbb{C}} - 2r]) \end{aligned}$$

and

$$\begin{aligned} &R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\, \mathbb{R}) \\ &\quad \simeq R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}+r-n}\, \mathbb{R}), \mathbb{R}[-2d_{\mathbb{C}}]) \\ &\qquad \simeq R\mathrm{Hom}(R\Gamma(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{d_{\mathbb{C}}+r-n}\, \mathbb{R}), \mathbb{R}[-2d_{\mathbb{C}}]). \end{aligned}$$

If $X(\mathbb{C})$ is not connected, we may apply the same argument to each connected component separately. This gives us (2.4.3). ■

2.4.4. Proposition.

0) *If $X = \coprod_{0 \le i \le r} X_i$ is a finite disjoint union of arithmetic schemes, then*

   0a) *the conjecture $\mathbf{L}^c(X_{\acute{e}t}, n)$ is equivalent to the conjunction of conjectures $\mathbf{L}^c(X_{i,\acute{e}t}, n)$ for $i = 0, \ldots, r$;*

   0b) *the conjecture $\mathbf{B}(X, n)$ is equivalent to the conjunction of conjectures $\mathbf{B}(X_i, n)$ for $i = 0, \ldots, r$.*

1) *If $U \hookrightarrow X \hookleftarrow Z$ is an open-closed decomposition, then*

   1a) *if two out of three conjectures $\mathbf{L}^c(U_{\acute{e}t}, n)$, $\mathbf{L}^c(Z_{\acute{e}t}, n)$, $\mathbf{L}^c(X_{\acute{e}t}, n)$ hold, then the other one holds as well;*

   1b) *if two out of three conjectures $\mathbf{B}(U, n)$, $\mathbf{B}(Z, n)$, $\mathbf{B}(X, n)$ hold, then the other one holds as well.*

2) For $r \geq 0$, consider the affine space $\mathbb{A}^r_X$:

$$\begin{tikzcd} \mathbb{A}^r_X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{A}^r_{\mathbb{Z}} \arrow[d] \\ X \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

2a) the conjectures $\mathbf{L}^c(\mathbb{A}^r_{X,\acute{e}t}, n)$ and $\mathbf{L}^c(X_{\acute{e}t}, n-r)$ are equivalent;

2b) the conjectures $\mathbf{B}(\mathbb{A}^r_X, n)$ and $\mathbf{B}(X, n-r)$ are equivalent.

Proof. Part 0) really deserved to be numbered by 0, because it is quite obvious: for finite disjoint unions $X := \coprod_{0 \leq i \leq r} X_i$ we have

$$R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \cong \bigoplus_{0 \leq i \leq r} R\Gamma(X_{i,\acute{e}t}, \mathbb{Z}^c(n)),$$

which implies 0a). Similarly, for 0b), we note that the regulator morphism and its dual decompose as

$$Reg_X \cong \bigoplus_{0 \leq i \leq r} Reg_{X_i} \quad \text{and} \quad Reg^\vee_X \cong \bigoplus_{0 \leq i \leq r} Reg^\vee_{X_i}.$$

As for open-closed decompositions, recall that in this situation we have a distinguished triangle (see 0.11.1)

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n))[1]$$

The associated long exact sequence in cohomology

$$\begin{aligned} \cdots \to H^i(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^i(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to H^i(U_{\acute{e}t}, \mathbb{Z}^c(n)) \\ \to H^{i+1}(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to \cdots \end{aligned}$$

implies 1a). For 1b), we apply $R\mathrm{Hom}(-, \mathbb{R})$ to the morphism of triangles from 2.2.2:

$$\begin{tikzcd} R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \arrow[r, "Reg^\vee_U"] \arrow[d] & R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \arrow[d] \\ R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \arrow[r, "Reg^\vee_X"] \arrow[d] & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \arrow[d] \\ R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \arrow[r, "Reg^\vee_Z"] \arrow[d] & R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \arrow[d] \\ R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \arrow[r, "Reg^\vee_U{[1]}"] & R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[1] \end{tikzcd}$$

Here if two of the arrows $Reg_U^\vee$, $Reg_X^\vee$, $Reg_Z^\vee$ is a quasi-isomorphism, the third one is also a quasi-isomorphism by the triangulated 5-lemma.

In 2), we have according to [Mor2014, Lemma 5.11] a quasi-isomorphism of complexes of sheaves on $X_{\acute{e}t}$

$$Rp_* \mathbb{Z}^c(n) \simeq \mathbb{Z}^c(n-r)[2r],$$

so that there is a quasi-isomorphism

$$R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)) \xrightarrow{\simeq} R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r]. \tag{2.4.4}$$

This establishes 2a). As for 2b), it follows from commutativity of the diagram from 2.2.3:

$$\begin{tikzcd}
R\Gamma_c(G_\mathbb{R}, \mathbb{A}^r_X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \arrow[r, "\cong"] \arrow[d, "Reg^\vee_{\mathbb{A}^r_X,n}"'] & R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})[-2r-1] \arrow[d, "Reg^\vee_{X,n-r}[-2r]"] \\
R\mathrm{Hom}(R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \arrow[r, "\cong"] & R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r)), \mathbb{R})[-2r]
\end{tikzcd}$$

Here the left vertical arrow is a quasi-isomorphism if and only if the right vertical arrow is a quasi-isomorphism. ∎

2.4.5. Theorem.

0) *If $X = \coprod_{0\le i\le r} X_i$ is a disjoint union of arithmetic schemes, then the conjectures $\mathbf{C}(X_i, n)$ for $i = 0, \ldots, r$ together imply $\mathbf{C}(X, n)$.*

1) *If $U \hookrightarrow X \hookleftarrow Z$ is an open-closed decomposition of an arithmetic scheme, then if two out of three conjectures $\mathbf{C}(U, n)$, $\mathbf{C}(Z, n)$, $\mathbf{C}(X, n)$ hold, the other one holds as well.*

2) *The conjecture $\mathbf{C}(\mathbb{A}^r_X, n)$ is equivalent to $\mathbf{C}(X, n-r)$.*

Proof. The conjecture $\mathbf{C}(X, n)$ has two different parts: one about the special value $\zeta^*(X, n)$ and the other one about the vanishing order of $\zeta(X, s)$ at $s = n$. For the special value part of the conjecture, the claim holds thanks to 2.4.1 and 2.4.2. The vanishing order part is actually easier, because it is just about counting ranks of cohomology groups.

In the view of (2.4.1) and (2.4.2), we have

$$\operatorname{ord}_{s=n} \zeta(X, s) = \operatorname{ord}_{s=n} \zeta(U, s) + \operatorname{ord}_{s=n} \zeta(Z, s)$$

and

$$\operatorname{ord}_{s=n} \zeta(\mathbb{A}^r_X, s) = \operatorname{ord}_{s=n-r} \zeta(X, s).$$

This means that 0), 1), 2) would follow respectively from the identities

(2.4.5)
$$\sum_{j\in\mathbb{Z}}(-1)^j\cdot j\cdot \mathrm{rk}_{\mathbb{Z}}\, H^j_{W,c}(X,\mathbb{Z}(n)) \stackrel{?}{=} \sum_{0\le i\le r}\sum_{j\in\mathbb{Z}}(-1)^j\cdot j\cdot \mathrm{rk}_{\mathbb{Z}}\, H^j_{W,c}(X_i,\mathbb{Z}(n)),$$

(2.4.6)
$$\begin{aligned}\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n)) &\stackrel{?}{=}\\ \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(U,\mathbb{Z}(n)) &+ \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(Z,\mathbb{Z}(n)),\end{aligned}$$

(2.4.7)
$$\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(\mathbb{A}^r_X,\mathbb{Z}(n)) \stackrel{?}{=} \sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n-r)).$$

As for (2.4.5), it is enough to revise the construction of Weil-étale complexes and note that

$$R\Gamma_{W,c}(\coprod_{0\le i\le r} X_i,\mathbb{Z}(n)) \cong \bigoplus_{0\le i\le r} R\Gamma_{W,c}(X_i,\mathbb{Z}(n)).$$

Alternatively, thanks to 2.3.4, we may rewrite (2.4.5) as

$$\chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) \stackrel{?}{=} \sum_{0\le i\le r}\chi(R\Gamma_c(G_{\mathbb{R}},X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R})),$$

which is evident, as Euler characteristic is additive with respect to direct sums of complexes:

$$\begin{aligned}\chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) &= \chi(R\Gamma_c(G_{\mathbb{R}},\coprod_{0\le i\le r} X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R}))\\ &= \chi(\bigoplus_{0\le i\le r} R\Gamma_c(G_{\mathbb{R}},X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R}))\\ &= \sum_{0\le i\le r}\chi(R\Gamma_c(G_{\mathbb{R}},X_i(\mathbb{C}),(2\pi i)^n\,\mathbb{R})).\end{aligned}$$

Similarly, (2.4.6) is equivalent to

$$\begin{aligned}&\chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) \stackrel{?}{=}\\ &\qquad\chi(R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\,\mathbb{R})) + \chi(R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{R})),\end{aligned}$$

which is now obviously true, being the additivity of the usual Euler characteristic for the distinguished triangle

$$\begin{aligned}&R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\,\mathbb{R}) \to R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})\\ &\qquad\to R\Gamma_c(G_{\mathbb{R}},Z(\mathbb{C}),(2\pi i)^n\,\mathbb{R}) \to R\Gamma_c(G_{\mathbb{R}},U(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[1]\end{aligned}$$

Similarly, the identity (2.4.7) is equivalent to

$$\chi(R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) \stackrel{?}{=} \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})).$$

The two complexes

$$R\Gamma_c(G_{\mathbb{R}}, \mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \quad \text{and} \quad R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})$$

are quasi-isomorphic according to (2.4.3), modulo the shift by $2r$, which is an even number, so it does not affect the Euler characteristic. ■

Similarly to the relation (2.4.2), for projective spaces $\mathbb{P}^r_X := \mathbb{P}^r_{\mathbb{Z}} \times X$ we have

$$\zeta(\mathbb{P}^r_X, s) = \prod_{0 \le i \le r} \zeta(X, s-i).$$

Note that this follows by induction from (2.4.1) and (2.4.2). For $r = 0$, this is trivial. For the induction step, assume that the above formula holds for $\mathbb{P}^{r-1}_X$. Then for $\mathbb{P}^r_X$ we may consider the open-closed decomposition

$$\mathbb{A}^r_X \hookrightarrow \mathbb{P}^r_X \hookleftarrow \mathbb{P}^{r-1}_X$$

and then

$$\begin{aligned}\zeta(\mathbb{P}^r_X, s) &= \zeta(\mathbb{A}^r_X, s) \cdot \zeta(\mathbb{P}^{r-1}_X, s)\\ &= \zeta(X, s-r) \cdot \prod_{0 \le i \le r-1} \zeta(X, s-i) = \prod_{0 \le i \le r} \zeta(X, s-i).\end{aligned}$$

Applying the same inductive reasoning, we immediately deduce from 2.4.5 the compatibility of our main conjecture with taking the projective space.

2.4.6. Corollary. *For each arithmetic scheme X, assume $\mathbf{C}(X, n-i)$ holds for $i = 0, \ldots, r$. Then $\mathbf{C}(\mathbb{P}^r_X, n)$ holds.*

Conclusion

The conjecture $\mathbf{C}(X, n)$ is known for some special cases, e.g. thanks to its equivalence to the Tamagawa number conjecture in case when X is proper and regular (see the remark 2.3.3). It is now possible to take these cases as an input, and then formally deduce $\mathbf{C}(X, n)$ for new schemes constructed using the operations of disjoint unions, open-closed gluing and affine bundles. Note that these operations allow us to obtain non-smooth schemes.

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Abstract

This work is dedicated to interpreting in cohomological terms the special values of zeta functions of arithmetic schemes.

This is a part of the program envisioned and started by Stephen Lichtenbaum (see e.g. Ann. of Math. vol. 170, 2009), and the conjectural underlying cohomology theory is known as Weil-étale cohomology. Later on Baptiste Morin and Matthias Flach gave a construction of Weil-étale cohomology using Bloch's cycle complex and stated a precise conjecture for the special values of proper regular arithmetic schemes at any integer argument $s = n$. The goal is to extend the above mentioned result and conjecture to special values of arbitrary arithmetic schemes (possible singular or non-proper) while restricting to the case $n < 0$.

Following the ideas of Flach and Morin, the Weil-étale complexes are defined for $n < 0$ for arbitrary arithmetic schemes, under standard conjectures about finite generation of motivic cohomology. Then it is stated as a conjecture how these complexes are related to the special values. For proper and regular schemes, this conjecture is equivalent to the conjecture of Flach and Morin, which also corresponds to the Tamagawa number conjecture.

We prove that the conjecture stated in this work is compatible with the decomposition of an arbitrary scheme into an open subscheme and its closed complement. We also show that this conjecture for an arithmetic scheme X at $s = n$ is equivalent to the conjecture for $\mathbb{A}^r_X$ at $s = n - r$, for any $r \geq 0$. It follows that, taking as an input the schemes for which the conjecture is known, it is possible to construct new schemes, possibly singular or non-proper, for which the conjecture holds as well. This is the main unconditional outcome of the machinery developed in this thesis.

Résumé

Ce travail est dédié à l'interprétation en termes cohomologiques des valeurs spéciales des fonctions zêta des schémas arithmétiques.

C'est une partie d'un programme envisagé et initié par Stephen Lichtenbaum (voir par ex. Ann. of Math. vol. 170, 2009), et la théorie cohomologique sous-jacente s'appelle la cohomologie Weil-étale. Plus tard, Baptiste Morin et Matthias Flach ont donné une construction de la cohomologie Weil-étale en utilisant les complexes de cycles de Bloch, et ont énoncé une conjecture précise pour les valeurs spéciales des schémas arithmétiques propres et réguliers, en tout entier $s = n$. Le but de cette thèse est de généraliser le résultat et la conjecture mentionnés ci-dessus aux valeurs spéciales des schémas arithmétiques arbitraires (éventuellement singuliers ou non-propres) lorsque l'on se restreint au cas $n < 0$.

Suivant les idées de Flach et Morin, les complexes Weil-étale sont définis pour $n < 0$ pour les schémas arithmétiques arbitraires, sous des conjectures standards sur la génération finie de la cohomologie motivique. Ensuite, il est énoncé comme une conjecture de quelle manière ces complexes sont liés aux valeurs spéciales. Pour les schémas propres et réguliers, cette conjecture est équivalente a la conjecture de Flach et Morin, qui correspond aussi à la conjecture du nombre de Tamagawa.

On prouve que la conjecture énoncée dans ce travail est compatible avec la décomposition d'un schéma arbitraire en un sous-schéma ouvert et son complémentaire fermé. On montre aussi que cette conjecture pour un schéma arithmétique X en $s = n$ est équivalente à cette même conjecture pour $\mathbb{A}^r_X$ en $s = n - r$, pour tout $r \geq 0$. Il suit que, en partant des schémas pour lesquels la conjecture est connue, on peut construire de nouveaux schémas, éventuellement singuliers ou non-propres, pour lesquels la conjecture est également vraie. C'est le principal résultat inconditionnel issu de la machinerie développée dans cette thèse.

Samenvatting

Dit werk is gewijd aan het interpreteren in cohomologische termen van de speciale waarden van zeta-functies van aritmetische schema's. Dit is deel van een programma voorgesteld en gestart door Stephen Lichtenbaum (zie bijvoorbeeld Ann. of Math. vol. 170, 2009), en de conjecturale cohomologietheorie staat bekend als Weil-étale cohomologie. Later gaven Baptiste Morin en Matthias Flach een constructie van Weil-étale cohomologie gebruikmakend van het cykelcomplex van Bloch, en stelden zij een precies vermoeden op voor de speciale waarden in willekeurige gehele getallen $s = n$ van zeta functies van propere reguliere aritmetische schema's. Het doel van dit proefschrift is het bovengenoemde resultaat en vermoeden uit te breiden naar speciale waarden van willekeurige aritmetische schema's (mogelijk singulier of niet-proper) onder de beperking dat $n < 0$.

In navolging van de ideeën van Flach en Morin definiëren we Weil-étale complexen voor $n < 0$ voor willekeurige aritmetische schema's, onder standaardvermoedens over eindige voortgebrachtheid van motivische cohomologie. Vervolgens formuleren we een vermoeden hoe deze complexen gerelateerd zijn een speciale waarden. Voor propere en reguliere schema's is dit vermoeden equivalent aan dat van Flach en Morin, dat ook correspondeert met het zogenaamde 'Tamagawa getal vermoeden'.

We bewijzen dat ons vermoeden compatibel is met decomposities van een willekeurig schema in een open deelschema en het gesloten complement ervan. We laten ook zien dat het vermoeden voor een aritmetisch schema X in $s = n$ equivalent is met het vermoeden voor $\mathbb{A}^r_X$ in $s = n - r$, voor elke $r \geq 0$. Daaruit volgt dat, vanuit schema's waarvoor het vermoeden bekend is, het mogelijk is nieuwe schema's, mogelijk singulier of niet-proper, te construeren waarvoor het vermoeden dan ook waar is. Dit is de het belangrijkste gevolg dat onafhankelijk is van vermoedens, van de in dit proefschrift ontwikkelde machinerie.

Curriculum Vitae

Alexey Beshenov was born in 1989 in Lipetsk, USSR. He studied there at the Liceum N. 44 in a class with specialization in mathematics and physics.

In 2006–2010 he obtained a bachelor degree in computer programming at the Lipetsk Polytechnic University.

In order to study mathematics, he moved to Saint Petersburg, where in 2010–2012 he finished the master program in theoretical computer science at the University of the Russian Academy of Sciences, curated by the Laboratory of Mathematical Logic of the St. Petersburg Department of Steklov Institute. Under supervision of Dmitriĭ Pasechnik he defended his thesis on algorithmic aspects of real algebraic geometry.

In 2012 Alexey was awarded an Erasmus Mundus scholarship to study at the University of Milan and University of Bordeaux within the ALGANT Master program, whose main focus is geometry and number theory. He obtained a master degree in pure mathematics in 2014 and defended his thesis on Borel's results on algebraic K-theory of number rings, advised by Boas Erez.

Again supported by an Erasmus Mundus scholarship, he did his PhD studies at the University of Bordeaux and Universiteit Leiden in 2014–2017 under direction of Baptiste Morin and Bas Edixhoven.