SEMINAR ON ÉTALE COHOMOLOGY OF NUMBER FIELDS

by

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1.)

Notation (1.1) k = a number field.

A = integers in k .

X = Spec A .

$U \subset X$ a nonempty Zariski-open subset.

The etale cohomology of U with values in the multiplicative group $\mathbb{G}_m$ can be described by class field theory as follows:

Denote by

$$i: \text{Spec } k \longrightarrow U$$

the map. One has the usual exact sequence

$$(1.2) \qquad 0 \longrightarrow (\mathbb{G}_m)_U \longrightarrow i_*(\mathbb{G}_m)_k \longrightarrow D \longrightarrow 0$$

where

$$D = \bigoplus_{\substack{x \text{ closed} \\ \text{in } U}} (\mathbb{Z})_x$$

is the sheaf of Cartier divisors on U . Now by local class field theory,

$$(1.3) \qquad R^q i_* \mathbb{G}_m = 0, \quad q > 0, \qquad \text{i.e.,}$$

$$H^q(U, i_*\mathbb{G}_m) \approx H^q(\text{Spec } k, \mathbb{G}_m) \ , \quad \text{all } q \ .$$

Taking into account the vanishing of certain groups, the exact cohomology sequence of (1.2) yields the exact sequences

$$(1.4) \qquad 0 \longrightarrow \mathbb{G}_m(U) \longrightarrow k^* \longrightarrow \bigoplus_x (\mathbb{Z}) \longrightarrow \text{Pic } U \longrightarrow 0 \ ,$$

$$0 \longrightarrow H^2(U, \mathbb{G}_m) \longrightarrow \text{Br } k \xrightarrow{\phi} \bigoplus_x (\mathbb{Q}/\mathbb{Z}) \longrightarrow H^3(U, \mathbb{G}_m) \longrightarrow 0$$

where $\mathrm{Br}\, k = H^2(\mathrm{Spec}\ k, \mathbb{G}_m)$ and $\mathbb{Q}/\mathbb{Z} = H^2(k(x), \mathbb{Z})$. The map ϕ is the one given by class field theory.

Corollary (1.5): If k is totally imaginary then

$$H^q(X,\mathbb{G}_m) = \begin{cases} A^* , & q = 0, \\ \mathrm{Pic}\ X , & q = 1, \\ 0 , & q = 2, \\ \mathbb{Q}/\mathbb{Z} , & q = 3. \end{cases}$$

Theorem (1.6): Suppose $p \neq 2$ or that k is totally imaginary. Then $cd_p X = 3$ and $cd_p U = 2$ if $U \neq X$.

2.)

In this section we denote by $f: X \longrightarrow \mathrm{Spec}\ \mathbb{Z}$ a scheme of finite type. Because of "Artin-Schreier" theory, one can show that for a scheme Y of characteristic p

(2.1) $$cd_p Y \leq \mathrm{cdqc}\ Y + 1 \qquad (p = \mathrm{char}\ Y)$$

where $\mathrm{cdqc}\ Y = \sup[q \mid H^q(Y,F) \neq 0$ for some quasi-coherent sheaf F on $Y]$. Using this and dimension theory for fields, one obtains

Theorem (2.2): $cd_p X \leq 2 \dim X + 1$ if $p \neq 2$.

The rest of this section is devoted to 2-cohomology.

Notation (2.3): X_∞ = space of closed points of $X \otimes_{\mathbb{Z}} \mathbb{R}$ with the real topology

= $X(\mathbb{C})/G$, where $X(\mathbb{C})$ is the space of points of X with values in $\mathbb{C}$, with the usual topology, and where $G = \mathbb{Z}/2$ operates by complex conjugation.

$X(\mathbb{R})$ = real locus of X, which is a closed subspace of X_∞.

$\bar{X}$ = the topological space whose underlying set is $X \amalg X_\infty$ with the topology whose open sets are pairs (X', U) where X' is a Zariski open set in X, and U is an open subset of X'_∞.

Actually, we will work with the following étale topology on $\bar{X}$: The category of open sets are pairs $(f: X' \longrightarrow X, U)$ consisting of a morphism of schemes f and an open subset U of X'_∞ having the following properties:

(a) f is étale.

(b) In the map $g: U \longrightarrow X_\infty$ induced by f, $g(u) \in X(\mathbb{R}) \Rightarrow u \in X'(\mathbb{R})$.

A map $(f_1, U_1) \longrightarrow (f_2, U_2)$ is a map $X'_1 \longrightarrow X'_2$ commuting with the structure maps and such that under the induced map $X'_{1\infty} \longrightarrow X'_{2\infty}$, U_1 is carried into U_2. A family of maps with range (f, U) is a covering iff (X', U) is the union of the images.

For this topology, there are morphisms $X_{et} \xrightarrow{j} \bar{X}_{et}$ and $X_\infty \xrightarrow{i} \bar{X}_{et}$ where X_∞ is taken with the topology of local isomorphisms. The map j is formally an open immersion and i is its closed complement. The derived functors $R^q j_* F$ for a sheaf F on X_{et} are 2-torsion sheaves concentrated on the real locus $X(\mathbb{R})$, $q > 0$.

Theorem (2.4): Let X = Spec A be the ring of integers in a number field, and set $(\mathbb{G}_m)_{\bar{X}} = j_*(\mathbb{G}_m)_X$. Then

$$H^q(\bar{X}, \mathbb{G}_m) = \begin{cases} A^*, & q = 0, \\ \text{Pic } X, & q = 1, \\ 0, & q = 2, \\ \mathbb{Q}/\mathbb{Z}, & q = 3, \\ 0, & q > 3 \end{cases}.$$

(Slight variations in dimensions 0,1 could be obtained by insisting that a unit of $\mathbb{G}_m$ be positive at a real prime.) The above is an easy consequence of the following theorem:

Theorem (Tate): Let k be a number field and F a sheaf on Spec k. Then

$$H^q(\text{Spec } k, F) \longrightarrow H^q(\text{Spec}(k \otimes_{\mathbb{Z}} \mathbb{R}), F_{\mathbb{R}})$$

is surjective, $q=2$, and bijective, $q > 2$. Here $F_{\mathbb{R}}$ denotes the induced sheaf.

<u>Theorem (2.5)</u>: Let F be a sheaf on $\bar{X}$ whose restriction to X is a noetherian torsion sheaf. Then $H^q(\bar{X}, F) = 0$ for $q > 2 \dim X + 1$.

<u>Corollary (2.6)</u>: (a) $H^q(X, F) \xrightarrow{\sim} H^q(X \otimes_{\mathbb{Z}} \mathbb{R}, F_{\mathbb{R}})$ for $q > 2\dim X + 1$.

(b) $cd_2 X < \infty \Leftrightarrow cd_2 X \leq 2 \dim X + 1 \Leftrightarrow X(\mathbb{R}) = \emptyset$.

(c) for a field K of finite type, $cd_2 K = \infty$ iff K is a real field.

(Part (c) is also an easy consequence of a general result of Serre.)

3)

We use the notations of section 1. Let $F^\cdot$ be a complex of sheaves over X whose cohomology is bounded (i.e., $H^q(F^\cdot) = 0$ for q sufficiently large) and such that $H^q(F^\cdot)$ is a noetherian torsion sheaf for all q.

We denote by $\underline{\underline{H}}^q(X, F^\cdot)$ the hypercohomology of X into $F^\cdot$ and by $\underline{\underline{\mathrm{Ext}}}^q(X; F^\cdot, G_m)$ the global hyper-Ext on X. For any q those groups are finite commutative groups and for q sufficiently large they are equal to zero.

For any prime integer p and for any finite commutative group M we denote by M_p the p-primary component of M.

<u>Theorem (3.1)</u>: The Yoneda product

$$(*)_p \quad \underline{\underline{H}}^q(X, F^\cdot)_p \times \underline{\underline{\mathrm{Ext}}}^{3-q}(X; F^\cdot, G_m)_p \longrightarrow H^3(X, G_m)_p \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

is a perfect duality for $p \neq 2$. If k is a totally imaginary field, the pairing $(*)_2$ is also a perfect duality.

Let now U be an open subscheme of X and $F^\cdot$ a complex of sheaves on U satisfying the same conditions as in the beginning of the section. The complex $F^\cdot_U$ will be the complex of sheaves on X obtained by extending the complex $F^\cdot$ by zero. We define $\underline{\underline{H}}^2_c(U, F^\cdot)$ (hypercohomology with compact support on U) by the equality:

$$\underline{\underline{H}}^q_c(U, F^\cdot) = \underline{\underline{H}}^q(X, F^\cdot_U).$$

Similarly, given any complex $G^\cdot$ of sheaves on U (whose cohomology is bounded), we define the groups $\underline{\underline{\mathrm{Ext}}}^q_c(U; F^\cdot, G^\cdot)$ (Hyper-Ext with compact support) in the following way: First we take an injective resolution $I(G^\cdot)$ of $G^\cdot$ (i.e., a morphism of complexes $\rho: G^\cdot \longrightarrow I(G^\cdot)$ into a complex whose objects are injective sheaves which induces an isomorphism on the sheaves of cohomology). Then we define the complex of sheaves on U : Rhom $(F^\cdot, I(G^\cdot))$ to be the single complex of sheaves on U of sheaf homomorphism of $F^\cdot$ into $I(G^\cdot)$. Then we define $\underline{\underline{Ext}}_c^q(U;F^\cdot,G^\cdot)$ by the equality:

$$\underline{\underline{Ext}}_c^q(U;F^\cdot,G^\cdot) = \underline{\underline{H}}^q(X, \underline{Rhom}(F^\cdot, I(G^\cdot))_U).$$

When the complex $G^\cdot$ is the single sheaf $\mathbb{G}_m$, the complex Rhom $(F^\cdot, I(\mathbb{G}_m))$ will be denoted by $D(F^\cdot)$.

As an immediate corollary of the theorem 3.1, we obtain:

Corollary 3.2: The Yoneda product

$$\underline{\underline{H}}_c^q(U,F^\cdot)_p \times \underline{\underline{Ext}}^{3-q}(U;F^\cdot,\mathbb{G}_m)_p \longrightarrow \underline{\underline{H}}_c^3(U,\mathbb{G}_m)_p \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

is a perfect duality for any prime p different from 2. If k is totally imaginary, it is also a perfect duality for p = 2.

Let us denote by Δ the canonical morphism of complexes

$$\Delta: F^\cdot \longrightarrow D(D(F^\cdot)).$$

Theorem 3.3: When the torsion of the cohomology sheaves of $F^\cdot$ is prime to the residual characteristics of U, the morphism Δ induces an isomorphism on the sheaves of cohomology.

As an immediate corollary of the theorem 3.3, we obtain:

Corollary 3.4: The Yoneda product

$$\underline{\underline{H}}^q(U,F^\cdot)_p \times \underline{\underline{Ext}}_c^{3-q}(U;F^\cdot,\mathbb{G}_m)_p \longrightarrow \underline{\underline{H}}_c^3(U,\mathbb{G}_m)_p \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

is a perfect duality for any complex $F^\cdot$ whose torsion of cohomology sheaves is prime to the residual characteristics of U and for any prime p different from 2. As usual, when k is a totally imaginary field, the restriction $p \neq 2$ can be omitted.