ALGEBRAIC GEOMETRY
537-568

THE MOVING LEMMA FOR HIGHER CHOW GROUPS

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0. Introduction

Let X be a quasi-projective algebraic k-scheme, where k is a field. Let $U \subset X$ be Zariski open, and write $Y = X - U$. Let $\mathcal{Z}(X, \cdot)$, $\mathcal{Z}(U, \cdot)$, $\mathcal{Z}(Y, \cdot)$ be the simplicial abelian groups whose homotopy computes the higher Chow groups [1]. Writing

$$\Delta^n = \operatorname{Spec}\left(k[t_0, \cdots, t_n]/\left(\sum t_i - 1\right)\right),$$

by definition $\mathcal{Z}(X, n)$ is the free abelian group on irreducible subvarieties of $X \times \Delta^n$ meeting all faces properly. We write

$$CH^r(X, n) = H_n(\mathcal{Z}^r(X, \cdot)).$$

One has a left-exact sequence

$$0 \to \mathcal{Z}(Y, \cdot) \to \mathcal{Z}(X, \cdot) \to \mathcal{Z}(U, \cdot)$$

where the right-hand arrow fails to be surjective because cycles meeting faces properly on $U \times \Delta^n$ can have closures on $X \times \Delta^n$ which do not. The purpose of this paper is to prove

Theorem (0.1) (Moving lemma). *The map $\mathcal{Z}(X,\cdot)/\mathcal{Z}(Y,\cdot) \to \mathcal{Z}(U,\cdot)$ is a homotopy equivalence.*

Corollary (0.2). *Assume $Y \subset X$ has pure codimension d, so cycles of codimension p on Y have codimension $d+p$ on X. Then there is a long exact sequence*

$$\cdots \to CH^p(Y, n) \to CH^{p+d}(X, n) \to CH^{d+p}(U, n)$$
$$\to \cdots \to CH^p(Y, 0) \to CH^{p+d}(X, 0) \to CH^{p+d}(U, 0) \to 0.$$

The moving lemma was claimed in [1], but A. Suslin pointed out that the proof given there was not correct. A key ingredient in the present proof is the work of M. Spivakovsky [7]. I am indebted to him for a very helpful conversation. I should mention also that, recently, M. Levine [6]

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has given an alternative proof of the main result in [1] (relating the higher Chow groups $\otimes \mathbf{Q}$ with graded pieces of the γ-filtration on K-theory $\otimes \mathbf{Q}$) which avoids the moving lemma altogether.

In outline, the proof of (0.1) is as follows. Let $S_0 = (\Delta^1)^n$ and let $p : Z \to S_0$ be a quasiprojective morphism. We assume $p^{-1}(S_0 - \bigcup\{\text{faces}\})$ is dense in Z. We consider a tower

$$S_N \to S_{N-1} \to \cdots \to S_0 \tag{0.3}$$

where each map is the blowup with center an intersection of *distinguished divisors*. Distinguished divisors on S_0 are just the codimension-one faces, and distinguished divisors on S_i are defined inductively to be the strict transforms of distinguished divisors on S_{i-1} together with the exceptional divisor. Let $\Lambda^n = \mathrm{Spec}(k[x_1, \cdots, x_n])$. At each vertex $v \in S_N$ there are distinguished coordinates, and hence a rational map $\Lambda^n_v = \Lambda^n \dashrightarrow S_N$ which maps a neighborhood of 0 isomorphically to a neighborhood of v in S_N. The compositions

$$\pi_v : \Lambda^n_v \dashrightarrow S_N \to S_0,$$
$$\pi = \coprod \pi_v : \amalg \Lambda^n_v \to S_0$$

are everywhere regular. We write

$$\pi^! Z = \text{Zariski closure of } \left(\coprod \Lambda^n_v \times_{S_0} p^{-1}\left(S_0 - \bigcup\{\text{faces}\}\right)\right) \quad \text{in } \coprod \Lambda^n_v \times_{S_0} Z.$$

The basic point (2.1.2) is that there exists a tower (0.3) such that $\pi^! Z$ meets faces properly, i.e. that the inverse image in $\pi^! Z$ of a codimension-d intersection of distinguished divisors on $\coprod \Lambda^n_v$ has codimension $\geq d$. (By definition, distinguished divisors on Λ^n are defined by setting coordinates equal to zero.) When $Z \hookrightarrow S_0$ is a divisor, this is a consequence of the main theorem in [7].

The rest of the argument consists essentially in constructing homotopies. The situation is complicated and difficult to summarize concisely. Rather than working with cycles on $X \times \Delta^n$, we project and work with varieties over Δ^n. Essentially, one first passes from a variety over Δ^n to a variety over $(\Delta^1)^n$ (cf. (4.1.3)). Then, by the process outlined above, one gets a variety over $\coprod \Lambda^n_v$. Finally, by subdividing the Λ^n_v, one returns to a variety over Δ^n. What must be shown is that this whole process is homotopic to the identity.

It is possible to use algebraic cycles to construct a candidate for the category of mixed Tate motives over a field F [2, 3]. This category arises as the category of finite-dimensional graded representations of a graded

(pro-) Lie algebra $\mathscr{L}_F = \mathscr{L}_{-1} \oplus \mathscr{L}_{-2} \oplus \cdots$. In §5, (0.1) is used to construct a theory of specialization, at least in equal characteristic, for these motives. Suppose F is the quotient field of a discrete valuation ring $\mathscr{O}$ with residue field k. We assume $\mathscr{O}$ is a k-algebra. $\mathscr{L}_{F,-1}$ is the pro-object dual to $F^\times \otimes \mathbf{Q}$, so the valuation $F^\times \to \mathbf{Z}$ gives a homomorphism of graded Lie algebras $\mathrm{val} : \mathbf{Q}_{-1} \to \mathscr{L}_F$. We define a specialization Lie algebra homomorphism depending on the choice of a uniformizing element $\pi \in F$, $\mathrm{sp}_\pi : \mathscr{L}_k \to \mathscr{L}_F$. The image of sp_π centralizes val. If M is a mixed Tate motive over F corresponding to a Lie algebra representation $\rho : \mathscr{L}_F \to \mathrm{End}(M)$, we write $\Psi_\pi(M)$ for the mixed Tate motive over k corresponding to $\rho \circ \mathrm{sp}_\pi$. The endomorphism $\rho(\mathrm{val}(1))$ commutes with $\mathrm{sp}_\pi(\mathscr{L}_k)$ and so induces an endomorphism of degree -1,

$$N : \Psi_\pi(M) \to \Psi_\pi(M)(-1).$$

A mixed Tate motive over $\mathrm{Spec}(\mathscr{O})$ should then be a triple

$$(M_F, M_k, w)$$

where M_F (resp. M_k) is a mixed Tate motive over F (resp. k), and

$$w : M_k \to \ker(N) \subset \Psi_\pi(M)$$

is a map of $\mathscr{L}_k$-modules.

I want to acknowledge considerable help from K. Kato and M. Saito in understanding specialization.

1. Moving by blowup

(1.1) The basic idea. In this section we describe the basic method of blowing up which is employed in the moving lemma. Let S be a smooth quasi-projective variety over a field k, and let $D_1, \cdots, D_n$ be smooth divisors on S. Assume all intersections $D_I = D_{i_1} \cap \cdots \cap D_{i_p}$ are transverse (or empty). Let $\pi : T \to S$ be the blowup of S along D_I. Define divisors $D'_1, \cdots, D'_{n+1}$ on T by taking D'_{n+1} to be the exceptional divisor and D'_i = strict transform of D_i, $i \leq n$. Note that the divisors D'_i on T have the same transversal intersection property as the D_i on S.

Let $\mathscr{B}_S$ be the full subcategory of the category of schemes over S whose objects are varieties U obtained from S by iterating this process a finite number of times. S is the final object in $\mathscr{B}_S$. Note that each U comes equipped with a finite set of smooth Cartier divisors $\delta_1, \cdots, \delta_N$ meeting transversally. We will refer to the δ_i as *distinguished divisors*. Intersections of distinguished divisors are called *faces*. Faces of dimension 0 are *vertices*.

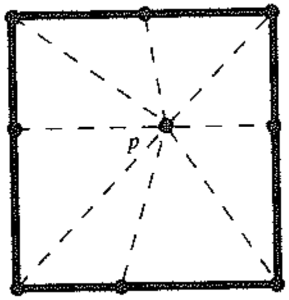

FIGURE 1

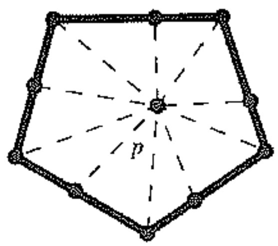

FIGURE 2

Let $f : X \to S$ be a quasiprojective morphism of schemes. We say f (or X) *meets faces properly* if for any face $\sigma \subset X$ of codimension r, $f^{-1}(\sigma)$ has codimension $\geq r$ in X. We always assume $f(X) \not\subset \bigcup\{\text{distinguished divisors}\}$. For $\pi : T \to S$ in $\mathscr{B}_S$, the *strict transform* $f' : X' \to T$ is the closure in $X \times_S T$ of

$$f^{-1}\left(S - \bigcup\{\text{distinguished divisors}\}\right).$$

One can show (using a theorem of Spivakovsky [7]) that for a suitable arrow π in $\mathscr{B}_S$, the strict transform f' meets faces properly.

Unfortunately, this property of the strict transform meeting faces properly is not quite what we want. To see the point, let $S = S_0 = \mathbf{A}^n$, with distinguished divisors obtained by setting coordinates equal to 0 or 1. This "cube" can be triangulated by choosing a general point $p \in S$ and mapping in

$$\Delta^n = \operatorname{Spec}(k[t_0, \cdots, t_n]/(t_0 + \cdots + t_n - 1))$$

in various ways suggested by Figure 1.

We can pullback our X over the various n-simplices to obtain a formal linear combination of schemes X_i over Δ^n. Suppose now we blow up a face of S (cf. Figure 2).

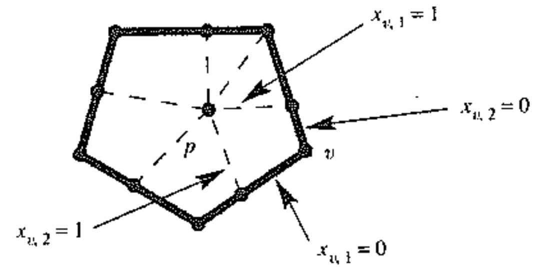

FIGURE 3

The resulting scheme T (illustrated in the example with a pentagon of distinguished divisors) no longer has a natural linear structure. The "obvious" triangulation illustrated by dotted lines in the figure cannot be defined algebro-geometrically. Instead we will show there are local coordinates $t_{1,v}, \cdots, t_{n,v}$ in the neighborhood of each vertex $v \in T$ such that the $t_{i,v}$ define the distinguished divisors in a neighborhood of v, and such that the rational map $\mu_v : \Lambda^n_v = \operatorname{Spec}(k[x_{1,v}, \cdots, x_{n,v}]) \dashrightarrow T$ gives an everywhere regular map

$$\pi_v : \Lambda^n_v \to S.$$

After suitably scaling the coordinates $x_{i,v}$ (with the scale factors depending on the original choice of $p \in S$) we arrive at the situation which is shown in Figure 3.

In words, if v and w are adjacent vertices in the sense that there exists a one-dimensional face of T containing v and w, then for suitable i and j the faces $x_{i,v} = 1$ and $x_{j,w} = 1$ can be identified as coordinatized schemes over S.

The key point of this construction is that, with suitable hypotheses, the formal linear combination of schemes over Δ^n obtained by triangulating S as above is "homotopic" to the linear combination obtained by pulling back first along the π_v with appropriate signs, and then triangulating the Λ_v and pulling back again over simplices in the triagulations. We will show that there exists $T \to S$ in $\mathscr{B}_S$ such that the strict transform of $f : X \to S$ along the map

$$\coprod_{\substack{v \in T \\ v \text{ vertex}}} \Lambda_v \to S$$

meets faces properly in a suitable sense. The combination of the homotopy result and the mechanism of having the strict transform meet faces properly leads to the proof of the moving lemma. We apologize for the necessity of mixing cubical and simplicial constructions. The point is that in the cubical theory one can achieve conveniently the necessary cancellation of interior faces as in Figure 3, while construction of the homotopy is easier in the simplicial theory.

(1.2) Properties of $\mathscr{B}_S$.

Lemma (1.2.1). *Let S, $\{D_1, \cdots, D_n\}$ and V, $\{\delta_1, \cdots, \delta_m\}$ be smooth quasi-projective varieties with collections of smooth divisors with all intersections transverse. Let $\pi : V \to S$ be a morphism (not necessarily in $\mathscr{B}_S$), and suppose $\pi^*(D_i) = \sum_j n_{ij}\delta_j$, $1 \le i \le n$. Let $\sigma : S' \to S$ be the blowup of $D_1 \cap D_2 \cap \cdots \cap D_r$. Then there exists an object $\tau : V' \to V$ in $\mathscr{B}_V$ and a diagram*

$$\begin{array}{ccc} V' & \xrightarrow{\tau} & V \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\sigma} & S \end{array}$$

The map π' has the property that, writing $\delta'_{i\nu}$ for the distinguished Cartier divisors on V' and D'_j for the distinguished divisors on S', we have

$$\pi'^*(D'_j) = \sum_{i,\nu} n_{ji\nu}\delta'_{i,\nu}.$$

Proof. Given $\tau : V' \to V$, the universal mapping property of blowups implies the map π' exists if and only if the pulllback

$$(\pi \circ \tau)^{-1}(\text{ideal defining } D_1 \cap \cdots \cap D_r)$$

is invertible. The same principle implies that $\sigma : S' \to S$ is dominated by a composition of blowups of intersections of two distinguished divisors. Indeed, if $f : T \to S$ is the blowup of $D_{r-1} \cap D_r$ and $E \subset T$ is the exceptional divisor,

$$f^{-1}(D_1 \cap \cdots \cap D_r) = f^{-1}(D_1) \cap \cdots \cap f^{-1}(D_{r-2}) \cap E.$$

We may thus argue by induction on r. It will therefore suffice to prove (1.2.1) in the case $r = 2$.

We must show then that by iterated blowups of intersections of distinguished divisors, starting from V, we can make the pullback of

$$\left(\sum n_{i1}\delta_i\right) \cap \left(\sum n_{i2}\delta_i\right)$$

locally principal. After each blowup step, we can remove the largest common Cartier divisor from the two sums and work with the *residual intersection* (cf. Fulton [4, Chapter 9]).

Consider first an intersection

$$(\delta) \cap (n_1\delta_1 + \cdots + n_\ell\delta_\ell)$$

where δ and the δ_i are distinct distinguished divisors. Blow $\delta \cap \delta_1$ and write e, $\overline{\delta}$, and $\overline{\delta}_1$ for the exceptional divisor and the two strict transforms. Since $\overline{\delta} \cap \overline{\delta}_1 = \varnothing$, the residual scheme on the blowup is

$$(\overline{\delta}) \cap ((n_1 - 1)e + n_2\delta_2 + \cdots + n_\ell\delta_\ell).$$

(We keep the same notation for a divisor and its pullback when the pullback coincides with the strict transform.) By induction on $\sum n_i$ we are done in this case.

Next consider

$$(n\delta) \cap (n_1\delta_1 + \cdots + n_\ell\delta_\ell).$$

Let $\delta : f = 0$ and $\sum n_i\delta_i : g = 0$ be local defining equations. Passing to a blowup and using the first case, we can assume $f = f'h$, $g = g'h$ with $(f', g') = (1)$. Thus

$$(f^n, g) = h \cdot (f'^n h^{n-1}, g') = h \cdot (h^{n-1}, g').$$

Again we conclude by induction, this time on n.

The general case is analogous, Changing notation slightly, suppose given

$$A = (n_1\delta_1 + \cdots + n_\ell\delta_\ell) \cap (p_1\epsilon_1 + \cdots + p_m\epsilon_m).$$

By induction on $\ell + m$, we may suppose after blowing up various intersections of pairs of distinguished divisors that the intersection

$$(n_2\delta_2 + \cdots + n_\ell\delta_\ell) \cap (p_1\epsilon_1 + \cdots + p_m\epsilon_m)$$

becomes a Cartier divisor. In terms of local defining ideals, we have

$$A : (x_1^{n_1}uf, vf)$$

where $(u, v) = (1)$. The residual intersection of A is then given locally by the ideal $(x_1^{n_1}, v)$ and we are back to the previous case. This proves the lemma. Q.E.D.

Corollary (1.2.2). *Given $f_i \to S$ in $\mathscr{B}_S$ for $1 \le i \le r$, there exists $h : W \to S$ in $\mathscr{B}_S$ and $g_i : W \to U_i$ with $f_i \circ g_i = h$ for $1 \le i \le r$.*

Proof. One reduces to the case $r = 2$, where it follows from noting that if $\pi : V \to S$ in (1.2.1) is in $\mathscr{B}_S$, then so is $\pi \circ \tau$. Q.E.D.

Remark (1.2.3). The category $\mathscr{B}_S$ is a full subcategory of the category of schemes over S, so it is not necessarily the case that a morphism in $\mathscr{B}_S$ is a succession of blowups. One way to think about the situation is to note that formally locally over S, objects in $\mathscr{B}_S$ can be given the structure of toric variety in such a way that faces are orbits and morphisms are compatible with the toric structure. As an application, we have

Lemma (1.2.4). *Let $f : T_1 \to T_2$ be a morphism in $\mathscr{B}_S$. Then f carries vertices to vertices.*

Proof. The problem is formal local over S, so we may assume the T_i are toric and f is compatible with the action. Vertices are then the fixed points of the torus action, so the assertion is clear. Q.E.D.

(1.3) Little cubes. We take $S = S_0 = \mathrm{Spec}(k[x_1, \cdots, x_n])$ with distinguished divisors defined by $x_i = 0, 1$. We consider a tower of blowups of faces

$$S_N \to \cdots \to S_1 \to S_0.$$

Let $v \in S_N$ be a vertex. By induction on N we define an open immersion of some Zariski open neighborhood of $0 \in \Lambda^n = \mathrm{Spec}(k[t_1, \cdots, t_n])$ onto an open neighborhood of $v \in S_N$,

$$\mu_v : \Lambda^n \supset U \hookrightarrow S_N ; \qquad \mu_v(0, \cdots, 0) = v$$

so that the distinguished divisors through v pull back to the coordinate hyperplanes $t_i = 0$. When $N = 0$ define the pullback on functions by

$$\mu_v^*(x_i) = \begin{cases} t_i, & \text{if } x_i(v) = 0; \\ 1 - t_i, & \text{if } x_i(v) = 1. \end{cases}$$

This defines μ_v for $N = 0$. Suppose now $N \geq 1$ and let $\overline{v}$ be the image of v in S_{N-1}. Let $\overline{t}_1, \cdots, \overline{t}_n$ be the coordinates at $\overline{v}$ defined inductively via $\mu_{\overline{v}}$. Over some neighborhood of $\overline{v}$, S_N is defined from S_{N-1} by blowing up $\{\overline{t}_i = 0 \mid i \in I\}$ for some (possibly empty) $I \subset \{1, \cdots, n\}$. If $I = \varnothing$, we take $t_i = \overline{t}_i$. Otherwise there will be a unique $i \in I$ such that v does not lie on the strict transform in S_N of $\{\overline{t}_i = 0\}$. Define

$$t_j = \begin{cases} \overline{t}_j, & \text{if } j \notin I; \\ \overline{t}_j/\overline{t}_i, & \text{if } j \in I \text{ and } j \neq i; \\ \overline{t}_i, & \text{if } j = i. \end{cases}$$

The coordinate system at a vertex v defined in this way will be called the *distinguished coordinate system* at v.

Lemma (1.3.1). (i) *The composition*

$$\pi_v : \Lambda^n \overset{\mu_v}{\dashrightarrow} S_N \to S$$

is everywhere regular.

(ii) *The assignment*

$$T \rightsquigarrow \coprod_{\substack{v \in T \\ v \text{ vertex}}} \Lambda^n$$

is a functor on $\mathscr{B}_S$.

Proof. These assertions are straightforward. Q.E.D.

By definition, an *edge* in S_N is a one-dimensional face.

Lemma (1.3.2). *Every edge in S_N contains exactly two vertices.*

Proof. This is clear for $N = 0$. For $N \geq 1$, let $\rho : S_N \to S_{N-1}$ and let $\ell \subset S_N$ be an edge. If $\rho(\ell)$ has dimension 1, then $\rho|\ell$ is an isomorphism, $\rho(\ell)$ is an edge of S_{N-1}, and the vertices of ℓ and $\rho(\ell)$ coincide under ρ. If $\rho(\ell)$ is a point, it is a vertex of S_{N-1}. The fibre $\rho^{-1}\rho(\ell)$ is then a projective space which is an intersection of distinguished divisors. The intersections of other distinguished divisors with $\rho^{-1}\rho(\ell)$ form the coordinate hyperplanes of the projective space. One is thus reduced to the situation where the ambient space is projective space, and the distinguished divisors are coordinate hyperplanes, where it is clear. Q.E.D.

Definition (1.3.3). Let v, w be the two vertices on an edge ℓ, and let $t_1, \cdots, t_n$ (resp. $u_1, \cdots, u_n$) be the distinguished coordinate system at v (resp. at w). The *coordinate for ℓ at v* is the unique t_p which does not vanish on ℓ. Suppose t_p (resp. u_q) is the coordinate for ℓ at v (resp. at w). We define $g = g(v, w)$ in the symmetric group $\mathscr{S}_n$ such that $g(p) = q$ and, for $i \neq p$, the distinguished divisors defined locally by $t_i = 0$ and $u_{g(i)} = 0$ coincide.

Our "little cubes" $\pi_v : \Lambda_v \to S$ are not yet really cubes. The sides $t_i = 0$ are defined, but we want to choose appropriate scale factors $c_{v,i}$ so the other sides are given by $t_i = c_{v,i}$. Choose once and for all a point

$$c = (c_1, \cdots, c_n) \in S - \bigcup\{\text{distinguished divisors}\}.$$

Note that $\pi_v^{-1} : S - \bigcup\{\text{distinguished divisors}\} \to \Lambda_v$ is defined. Let

$$c_v = (c_{v,1}, \cdots, c_{v,n}) \in \Lambda_v$$

be the image of c.

Lemma (1.3.4). *Let the situation and notation be as in* (1.3.3). *Define divisors $D_v = \{t_p = c_{v,p}\} \subset \Lambda_v^n$ (resp. $D_w = \{u_q = c_{w,q}\} \subset \Lambda_w^n$). Note D_v (resp. D_w) has coordinates t_α, $\alpha \neq p$ (resp. u_β, $\beta \neq q$). We have D_v isomorphic to D_w over S, the isomorphism carrying t_α to $u_{g(v,w)(\alpha)}$.*

Proof. To see that the two divisors are isomorphic over S, we argue by induction on N. When $N = 0$ the assertion is immediate. Suppose now that $N \geq 1$. Assume first that the image of the edge ℓ in S_{N-1}, $\overline{\ell} \subset S_{N-1}$ is an edge with vertices $\overline{v}$ and $\overline{w}$. We have $\Lambda_v^n \to \Lambda_{\overline{v}}^n$ (resp. $\Lambda_w^n \to \Lambda_{\overline{w}}^n$). The divisors D_v and D_w are the inverse images of the corresponding divisors $D_{\overline{v}}$ and $D_{\overline{w}}$ on $\Lambda_{\overline{v}}^n$ and $\Lambda_{\overline{w}}^n$. By induction, $D_{\overline{v}} \cong D_{\overline{w}}$, the isomorphism carrying (with obvious notation) coordinates $\overline{t}_\alpha$, $\alpha \neq p$, into coordinates $\overline{u}_\beta$, $\beta \neq q$, up to a permutation of indices. The origin in these coordinates is where the image of the edge $\overline{\ell}$ on $\Lambda^n_{\overline{v}}$ (resp. on $\Lambda^n_{\overline{w}}$) meets $D_{\overline{v}}$ (resp. $D_{\overline{w}}$). Further, D_v and D_w are coordinate patches on the blowup of $D_{\overline{v}}$ and $D_{\overline{w}}$ along the intersection of the center for the blowups on the Λ's with these divisors. The coordinate patches coincide, so $D_v \cong D_w$ as claimed.

It remains to consider the case when the image of the edge ℓ in S_{N-1} is a single vertex z. Let $y_1, \cdots, y_n$ be the coordinates at z. Let $\{y_i = 0 | i \in I\}$ define the center of the blowup $S_N \to S_{N-1}$. Then for some $p, q \in I$, the coordinates at v (resp. w) are y_i/y_p, $i \in I$, together with y_p and y_j, $j \notin I$ (resp. y_i/y_q, $i \in I$, together with y_q and y_j, $j \notin I$). We have

$$D_v : y_q/y_p = c_{z,q}/c_{z,p} \text{ (resp. } D_w : y_p/y_q = c_{z,p}/c_{z,q}).$$

The identification in question arises from the natural isomorphism

$$k[y_i/y_p, y_p, y_j]/(y_q/y_p - c_{z,q}/c_{z,p}) \cong k[y_i/y_q, y_q, y_j]/(y_p/y_q - c_{z,p}/c_{z,q})$$

which sends $y_p \mapsto (c_{z,p}/c_{z,q})y_q$. Q.E.D.

2. The basic general position theorem

(2.1) Formulation of the theorem. This section, which contains the heart of the proof of the moving lemma, is, strictly speaking, a bit out of place. We should first push the ideas in §1 further in order to construct a homotopy. However, this involves some rather fussy business with signs. Psychologically, it seems important now to give the reader a glimpse of the light at the end of the tunnel.

Let $S = \mathrm{Spec}(k[x_1, \cdots, x_n])$ with distinguished divisors $x_i = 0, 1$ as usual. Let $\pi : T \to S$ and $p : Z \to S$ be morphisms of schemes. Recall the strict transform of Z, $\pi^! Z$, is the closure in $Z \times_S T$ of $p^{-1}(S - \cup\{\text{distinguished divisors}\})$. Recall further that Z is said to meet faces properly if $p^{-1}(\sigma)$ has codimension $\geq r$ for every codimension r face σ on S.

Lemma (2.1.1). (i) *Let $f : T \to S$ be a morphism in $\mathscr{B}_S$. If $p : Z \to S$ meets faces properly, so does $f^! Z \to T$.*

(ii) *Let $\pi : \coprod \Lambda^n_v \to S$ be defined as in (1.3.1), where v runs through all vertices of T. For c a k-point of S not lying on any distinguished divisor, let c_v be the unique point of Λ^n_v lying over c. Define distinguished divisors on Λ^n_v to be the loci $t_{v,i} = 0$, $c_{v,i}$. Assume Z meets faces properly. Then there exists a proper closed subset $W \subsetneq S$ such that if $c \notin W$ then $\pi^! Z$ meets faces properly over $\coprod \Lambda^n_v$. In particular, if the ground field k is infinite, we can choose such a c (depending on Z).*

Proof. We will only prove (i). The argument for (ii) is the same, together with some standard general position arguments to handle the faces $t_{v,i} = c_{v,i}$. Let $q : f^! Z \to T$. For $\sigma \subset T$ a face of codimension d, we have $q^{-1}(\sigma)$ closed in $Z \times_S \sigma$, so it suffices to show $\dim(Z \times_S \sigma) \leq \dim Z - d$. For $r \geq 0$ let $S_r \subset S$ be the locally closed set where the fibres of p have dimension $\geq \dim(Z) - n + r$. Assume inductively that $q^{-1}(\sigma')$ has dimension $\leq \dim(Z) - e$ for any face $\sigma' \subsetneq \sigma$ with $e = \operatorname{codim} \sigma'$. Let $\partial\sigma \subset \sigma$ be the union of all proper subfaces of σ. Since $f | \sigma - \partial\sigma$ has equidimensional fibres, it will suffice to show that $f(\sigma) \cap S_r$ has codim $\geq r$ in $f(\sigma)$ for every r. By hypothesis, Z meets faces properly, so $\tau \cap S_r$ has codim $\geq r$ for any face $\tau \subset S$. When $d = n$ (so $\sigma = \text{point}$), $f(\sigma)$ is a face, so $f(\sigma) \cap S_r$ has codim $\geq r$. By induction, we may assume $f(\partial\sigma) \cap S_r$ has codim $\geq r$. To show the intersection $f(\sigma) \cap S_r$ has the correct dimension we remark that because of the existence locally of torus actions (1.2.3) we know that $f : \sigma - \partial\sigma \to S$ has image a dense subset of a face τ in S and has equidimensional fibres. Since $\tau \cap S_r$ has the correct dimension, the assertion follows. Q.E.D.

Theorem (2.1.2). *Let $p : Z \to S$ be quasi-projective. Assume the ground field k is infinite, and*

$$p(Z_i) \not\subset \bigcup\{\textit{distinguished divisors}\}$$

for every irreducible component $Z_i \subset Z$. Then there exists $T \to S$ a composition of blowups of faces such that, writing $\pi : \coprod \Lambda^n_v \to S$ for the corresponding map of local cubes and choosing $c \in S$ general, we have that $\pi^! Z$ meets faces properly over $\coprod \Lambda^n_v$.

Proof. One reduces easily to the case p projective and Z irreducible. We focus first on the case $P : Z \hookrightarrow S$ is a Cartier divisor. Consider the profinite set

$$\mathscr{V} = \varprojlim_{V \in \mathrm{Ob}(\mathscr{B}_S)} \{\text{vertices } \in V\}.$$

For $V \in \mathrm{Ob}(\mathscr{B}_S)$ let

$$\mathrm{pr}_V : \mathscr{V} \to \{\text{vertices of } V\}.$$

The main result in [7] has as an easy consequence the

Lemma (2.1.2.1). *Given $v \in \mathscr{V}$, there exists $c : V \to S$ a composition of blowups of faces such that $\mathrm{pr}_V(v) \notin c^! Z$.*

Proof of lemma. The question is local about a given vertex in S, so we may assume $Z : f(x_1, \cdots, x_n) = 0$ with vertex $(0, \cdots, 0)$. Consider in $\mathbb{R}^n_{\geq 0}$ the set M of points $(r_1, \cdots, r_n)$ such that the monomial $\prod x_i^{r_i}$ occurs in f with nonzero coefficient. Let $\triangle \subset \mathbb{R}^n_{\geq 0}$ be the convex hull of $\cup_{r \in M}(r + \mathbb{R}^n_{\geq 0})$. A subset $\Gamma \subset \{1, \cdots, n\}$ is chosen and a point $i \in \Gamma$ is given. The transformation

$$r'_j = \begin{cases} r_j, & \text{if } j \neq i; \\ -1 + \sum_{j \in \Gamma} r_j, & \text{if } i = j \end{cases}$$

is performed on $\triangle$, yielding a new convex set $\triangle'$. This is viewed as one "play" in a two-person game (Hironaka's game), with one player choosing the Γ and the other choosing the i. Spivakovsky shows [7, p. 420, theorem] there exists a strategy for the Γ player to "win" in the sense that, eventually, the convex set will contain a point with $\sum r_\alpha \leq 1$.

Geometrically, Γ corresponds to an ideal $\mathscr{I}(\Gamma) = \{x_j | j \in \Gamma\}$, and i is a vertex in the blowup. At that vertex, the coordinates are x_j/x_i for $j \in \Gamma$ and x_p for $p \notin \Gamma$. In these coordinates the convex set corresponding to $f(x_1, \cdots, x_n)/x_i$ is $\triangle'$. It may happen that the transform of f is divisible by x_i^ℓ for $\ell > 1$. In this case, the Γ player would choose $\Gamma = \{i\}$ (which has the effect of dividing f by x_i) repeatedly until this was no longer the case. The Γ strategy corresponds to the choice of $c : V \to S$ and the i strategy is the $v \in \mathscr{V}$.

Note that a point with $\sum r_\alpha \leq 1$ means that the divisor has multiplicity at most 1 at the vertex. Reordering the variables, we can write for some $1 \leq r \leq n$

$$f(x_1, \cdots, x_n) = x_1 g_1 + \cdots + x_r g_r + h(x_{r+1}, \cdots, x_n).$$

Here the g_i are polynomials in $x_1, \cdots, x_n$ with nonzero constant term and all terms in h have degree ≥ 2. If $h = 0$ (e.g. if $r = n$) we blow the ideal $(x_1, \cdots, x_r)$ and find that the strict transform does not pass through any vertex lying over $(0, \cdots, 0)$. If $h \neq 0$, we proceed by descending induction on r. We play Hironaka's game on h, modifying our technique so every time Spivakovsky's strategy suggests blowing h along an ideal generated by $\{x_\gamma | \gamma \in \Gamma\}$ we actually blow f along the ideal generated by $\{x_\gamma | \gamma \in \Gamma \cup \{1, \cdots, r\}\}$. Note that if the second player chooses $i \in \{1, \cdots, r\}$ we are done since the resulting polynomial (strict transform of f) does not vanish at the vertex. If the second player consistently chooses $i \geq r + 1$ we are essentially playing the Hironaka game on h. (The g_j change after every play, but they continue to have the same constant terms.) Eventually, then, by Spivakovsky's theorem, h acquires a nonzero linear term. At this point the linear term in f involves at least $r + 1$ variables, so we conclude by induction. Q.E.D.

Returning to the proof of (2.1.2), we continue to assume that Z is a Cartier divisor on S. The lemma implies $\mathrm{pr}_W(v) \notin g^! Z$ for any $g : W \to V \xrightarrow{c} S$ in $\mathscr{B}_S$. For each $v \in \mathscr{V}$, choose $V(v)$ in $\mathscr{B}_S$ so the strict transform of Z on $V(v)$ does not contain $\mathrm{pr}_{V(v)}(v)$. Let

$$U_v = \{\mu \in \mathscr{V} | \mathrm{pr}_{V(v)}(v) = \mathrm{pr}_{V(v)}(\mu)\}.$$

A finite number, say $U_{v_1}, \cdots, U_{v_h}$, of these open sets cover the compact set $\mathscr{V}$. By (1.2.2) we can find W in $\mathscr{B}_S$ dominating $V(v_1), \cdots, V(v_h)$. Any $\mu \in \mathscr{V}$ lies in one of the U_{v_j} which means $\mathrm{pr}_W(\mu)$ lies over $\mathrm{pr}_{V(v_j)}(v_j)$ so the strict transform of f on W does not vanish at $\mathrm{pr}_W(\mu)$. Since μ was arbitrary, it follows that the strict transform of f on W does not vanish at any vertex of W. Taking $S_N = W$ and noting that since Z is a Cartier divisor and the point $c \in S$ is general, $\pi^! Z$ meets faces properly on $\coprod \Lambda^n_v$ if and only if it does not contain any vertex $(0, \cdots, 0) \in \Lambda^n_v$, we have proved (2.1.2) in this case.

We next consider the case $p : Z \hookrightarrow S$ a closed immersion of codimension $d \geq 2$. We take $Z \subset W \subset S$ with W of codimension $d - 1$. By induction on d, we may assume $\pi^! W$ meets faces properly on $\coprod \Lambda^n_v$. Since $\pi^! Z \subset \pi^! W$, we can assume $\pi^! Z$ meets faces in codimension $\geq d - 1$. In particular, $\pi^! Z$ does not contain any vertices.

Let the above π correspond to $S' \to S$. We want to localize our problem around the vertices of S'. Suppose for each vertex $v \in S''$ we have an iterated sequence of blowups of intersections of distinguished divisors $T_v \to S'$ such that, for any vertex w on T_v lying over v, the strict transform of $\pi^! Z$ on Λ^n_w meets faces properly. Using (1.2.2) we may choose $T \to S'$ in $\mathscr{B}_S$ dominating all the T_v. For $x \in T$ a vertex, let $v \in S'$ and $w \in T_v$ lie under x. We have $\Lambda_x \to \Lambda_w \to \Lambda_v$. By assumption, the strict transform of $\pi^! Z$ on Λ_w meets faces properly, so by (2.1.1), the strict transform of $\pi^! Z$ on Λ_x does also. We conclude in this case that $\theta^! Z$ meets faces properly, where

$$\theta : \coprod_{x \in T} \Lambda_x \to S.$$

As a consequence of this discussion, we may replace $Z \subset S'$ with $Z \subset$

Λ^n where $\Lambda^n = \mathrm{Spec}(k[t_1, \cdots, t_n])$ with distinguished divisors $t_i = 0$. We may assume Z does not contain the origin, and our problem is to find $V \to \Lambda^n$ an iterated blowup of intersections of distinguished divisors such that, writing

$$\pi : \coprod_{v \in V} \Lambda^n_v \to \Lambda^n, \tag{2.1.2.2}$$

we have that $\pi^! Z$ meets faces properly. To this end, we consider a diagram

$$\begin{array}{ccccccc} Q_M & \to & Q_{M-1} & \to & \cdots & \to Q_0 & \xrightarrow{a} & \Lambda^n \\ \downarrow & & \downarrow & & & \downarrow b & & \\ P_M & \to & P_{M-1} & \to & \cdots & \to P_0 & & \end{array}$$

where $a : Q_0 \to \Lambda^n$ is the blowup of the origin on Λ^n, $P_0 = \mathbb{P}^{n-1}$, and the vertical arrow is induced by projection from the origin on Λ^n. As usual, the tower $P_M \to \cdots \to P_0$ consists of iterated blowups of intersections of distinguished divisors, where the coordinate hyperplanes are the distinguished divisors on P_0. $Q_i = P_i \times_{P_0} Q_0$, and the tower $Q_M \to \cdots \to \Lambda^n$ is also such an iterated blowup. Let $W \subset P_0$ be the closure of $b(a^{-1}(Z))$. W has codimension $d-1$, so by induction, we may assume for any vertex $v \in P_M$, the strict transform of W in Λ^{n-1}_v meets faces properly. Since $0 \notin Z$, the same will hold for the strict transform of Z in Λ^n_w for any vertex $w \in Q_M$. Taking $V = Q_M$, we get the desired diagram (2.1.2.2).

Finally we consider $p : Z \to S$ arbitrary projective. Applying what we have already proved to the closed subscheme $f(Z) \subset S$, we may assume that Z is irreducible and $f(Z) \subset S$ meets faces properly. We proceed by induction on $\dim(Z)$. If $d \leq 1$, it is straightforward to check that Z meets faces properly if and only if $f(Z)$ does, so we may assume $d \geq 2$.

Let $Z \subset \mathbb{P}^N \times S$ with $N \gg 0$, and fix an embedding $\psi : S \hookrightarrow (\mathbb{P}^N)^\vee$ (space of linear forms on $\mathbb{P}^N$). Let $F = k(\mathrm{GL}_{N+1})$ be the function field of the linear group, and let $g \in \mathrm{GL}_{N+1}$ be the generic point. Let $\kappa = g \circ \psi : S \hookrightarrow (\mathbb{P}^N)^\vee$, and let $\mathscr{H} \subset \mathbb{P}^N_F \times S$ be the corresponding divisor. Note $\mathscr{H}(\overline{k}) = \varnothing$. Indeed, if $h \in \mathscr{H}(\overline{k})$, let $H = \kappa(\mathrm{pr}_2(h))$ be the corresponding hyperplane. We have $\mathrm{pr}_1(h) \in H(\overline{k})$. On the other hand, $\kappa(\mathrm{pr}_2(h)) = g \circ \psi(\mathrm{pr}_2(h))$ is $\overline{k}$-generic, so $H(\overline{k}) = \varnothing$.

Let $Y = Z \cap \mathscr{H}$ and let $\rho : Y \to S$. Y is defined over F and $\dim_F(Y) = d - 1$. By induction, there exists $c : V \to S$ in $\mathscr{B}_S$ such that $\pi^! Y$ meets faces properly where

$$\pi : \coprod_{\substack{v \in V \\ v \text{ vertex}}} \Lambda^n \to S.$$

I claim that $\pi^! Z$ meets faces properly as well. The image of $\pi^! Z$ in $\coprod_{v \in V} \Lambda^n$ is $\pi^!(f(Z))$, which meets faces properly by (2.1.1), so if $\sigma \subset V$ is a bad face, there must be an irreducible component $T \subset \pi^!(f^{-1}(\sigma))$ such that the fibre dimension of T over σ is ≥ 1 and $\dim(T) > \dim(\sigma) + d - n$. Assume for a moment that

$$\pi^! Z \cap \left(\mathscr{H} \times_S \coprod \Lambda^n\right) = \pi^! Y. \tag{2.1.2.3}$$

Then $T \cap (\mathscr{H} \times_S V) \subset \pi^!(f^{-1}\sigma)$ so

$$\dim_F(\pi^!(f^{-1}\sigma)) \geq \dim_{\overline{k}}(T) - 1 > \dim(\sigma) + d - 1 - n,$$

contradicting the assumption that $\pi^!(Y)$ meets faces properly.

It remains to verify (2.1.2.3). The right-hand side is included in the left, so it suffices to show the part of the left side over any Λ^n is F-irreducible. It is clear that the left-hand side is a divisor on $\pi^! Z$ which coincides with $\pi^! Y$ over a dense open set of the given Λ^n. Let R be another component of this divisor. Let $\delta_1, \cdots, \delta_n$ be the distinguished divisors on Λ^n. We have

$$R \subset \pi^!\left(f^{-1}\left(\bigcup \delta_i\right)\right)$$

from which it follows that R is actually defined over $\overline{k}$. But $R(\overline{k}) \to \mathscr{H}(\overline{k}) = \varnothing$, so $R = \varnothing$. This completes the proof of Theorem (2.1.2). Q.E.D.

3. The homotopy

(3.1) Fussy signs. We fix a tower

$$S_N \to \cdots \to S_1 \to S$$

with $S = \mathbf{A}^n$ with distinguished divisors defined by $t_i = 0, 1$, and $S_{i+1} \to S_i$ given by blowing up a face for all i. We have the associated map of little cubes

$$\coprod_{\substack{v \in S_N \\ v \text{ vertex}}} \Lambda^n_v \to S.$$

Definition (3.1.1). For $w \in S$ a vertex we associate a sign $\epsilon(w) = (-1)^m$ where m is the number of coordinates taking the value 1 at w. For a vertex $v \in S_N$, $\epsilon(v) = \epsilon(w)$ where w is the image of v in S.

Let ℓ be an edge in S_N with vertices v and w, and let $g = g(v, w) \in \mathscr{S}_n$ (cf. (1.3.2) and (1.3.3)).

Lemma (3.1.2). $\epsilon(v) = -\epsilon(w)\,\mathrm{signature}(g)$.

Proof. Induction on N. When $N = 0$, $g = \mathrm{id}$ and $\epsilon(v) = -\epsilon(w)$. Assume $N \geq 1$ and that the lemma holds on S_{N-1}. Let $\rho : S_N \to S_{N-1}$ be the given map, which we take to be the blowup with center $Z \subset S_{N-1}$. We consider four cases:

Case 0. $\rho(v) \neq \rho(w)$; neither lie on Z. Then $g(v, w) = g(\rho(v), \rho(w))$ and we are done by induction.

Case 1. $\rho(v) \neq \rho(w)$; $\rho(v) \in Z$ and $\rho(w) \notin Z$. The distinguished divisors containing ℓ on S_N are the strict transforms of those containing $\rho(\ell)$ on S_{N-1}. Again, therefore, $g(v, w) = g(\rho(v), \rho(w))$ and we can use induction.

Case 2. $\rho(v) \neq \rho(w)$; $\rho(v)$ and $\rho(w) \in Z$. Exactly one of the distinguished divisors D defining Z on S_{N-1} will have strict transform D' not containing ℓ. Let $\bar{t}_1, \cdots, \bar{t}_n$ (resp. $\bar{u}_1, \cdots, \bar{u}_n$) be the distinguished coordinates at $\rho(v)$ (resp. $\rho(w)$) and assume D defined locally by $\bar{t}_i = 0$ (resp. $\bar{u}_{g(i)} = 0$). Let $J \subset \{1, \cdots, \}$ be such that $Z : \bar{t}_j = 0$, $j \in J$. The distinguished coordinates at v (resp. at w) will then be

$$t_r = \bar{t}_r,\ \ r \notin J;\ \ t_j = \bar{t}_j/\bar{t}_i \text{ if } j \in J - \{i\};\ \ t_i = \bar{t}_i$$

$$(\text{resp. } u_r = \bar{u}_r,\ \ r \notin g(J);\ \ u_s = \bar{u}_s/\bar{u}_{g(i)} \text{ if } s \in g(J - \{i\});\ \ u_{g(i)} = \bar{u}_{g(i)}).$$

Once again we will have $g(v, w) = g(\rho(v), \rho(w))$, the strict transform D' of D being replaced as a defining divisor for ℓ by the exceptional divisor E defined locally by $t_i = 0$ (resp. by $u_{g(i)} = 0$).

Case 3. $\rho(v) = \rho(w) = x \in Z$. Let $x_1, \cdots, x_n$ be the distinguished coordinates at x. Let $Z : x_j = 0$, $j \in J$. For some $p \in J$ (resp. $q \in J$) the coordinates at v (resp. w) are

$$t_r = x_r,\ \ r \notin J;\ \ t_j = x_j/x_p,\ \ j \in J - \{p\};\ \ t_p = x_p$$

$$(\text{resp. } u_r = x_r,\ \ r \notin J;\ \ u_j = x_j/x_q,\ \ j \in J - \{q\};\ \ u_q = x_q).$$

It follows that $g(v, w)$ is the transposition $g(p) = q$, $g(q) = p$, $g(i) = i$ for $i \neq p, q$. We have $\mathrm{signature}(g) = -1$ and $\epsilon(v) = \epsilon(w)$. Q.E.D.

Lemma (3.1.3). *Let $g \in \mathscr{S}_n$. Let $r \in \{1, \cdots, n\}$ and let $s = g(r)$. Let $\hat{g} \in \mathscr{S}_{n-1}$ be the composition*

$$\{1, \cdots, n-1\} \to \{1, \cdots, \hat{r}, \cdots, n\}$$
$$\to \{1, \cdots, \hat{s}, \cdots, n\} \to \{1, \cdots, n-1\},$$

where the two outside arrows are order preserving, and the map in the middle is induced by g. Then $\mathrm{signature}(\hat{g}) = (-1)^{r+s}\,\mathrm{signature}(g)$.

Proof. Straightforward. Q.E.D.

Lemma (3.1.4). *The signature of the cyclic permutation $g(p) = n$, $g(i) = i - 1$ for $p + 1 \leq i \leq n$, $g(j) = j$ for $j < p$, is $(-1)^{n-p}$.*

Proof. Straightforward. Q.E.D.

subdivision (Δ^2)

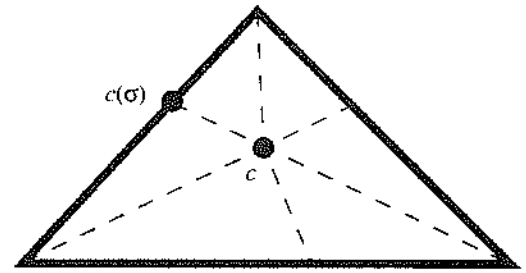

FIGURE 4

(3.2) Subdivision. The purpose of this section is to develop a general technique of subdivision, applicable to simplices and to products of simplices. The pictures are simple enough (cf. Figures 4 and 5) but one must be careful. In algebraic geometry, unlike in topology, faces of simplices extend infinitely. The fact that the large triangle in Figure 4 is the sum of the six interior triangles has to be properly understood.

We fix N a large integer and

$$c = (c_0, \cdots, c_N) \in \Delta^N = \mathrm{Spec}\left(k[t_0, \cdots, t_N]/\left(\sum t_i - 1\right)\right).$$

We assume $c_I = \sum_{i \in I} c_i \neq 0$ for any $I \subset \{0, \cdots, N\}$. Let

$$\sigma : t_j = 0,\ \ j \in J$$

be a simplex in Δ^N. Let $\{0, \cdots, N\} = I \amalg J$, and define

$$c(\sigma) = (\cdots, c_i/c_I, \cdots)_{i \in I} \in \sigma,$$

where the jth coordinate is 0 if $j \notin I$. Note $c(\{i\}) = c(i)$ is the ith vertex of Δ^N.

By a *general* p-simplex in Δ^N, we mean an affine linear map $\Delta^p \to \Delta^N$. Such a map is determined by the choice of $p + 1$ points $a(0), \cdots, a(p) \in \Delta^N$, and is denoted $\langle a(0), \cdots, a(p)\rangle$. For an *ordinary* p-simplex, we would require the $a(i)$ to be vertices of Δ^N. We also use the notation

$$\langle a(0), \cdots, a(p)\rangle = \langle a(0), \cdots, a(p-1)\rangle \llcorner a(p).$$

The boundary of a p-simplex is the formal linear sum

$$\partial\langle a(0), \cdots, a(p)\rangle = \sum (-1)^i \langle a(0), \cdots, \widehat{a(i)}, \cdots, a(p)\rangle.$$

We define a function

H_c: {formal linear sums of ordinary p-simplices in Δ^N} $\to$

{formal linear sums of general $(p+1)$-simplices in Δ^N}

for $0 \leq p \leq N$. (We frequently omit the subscript c when no confusion is likely.) For $p=0$ set

$$H_c(c(i)) = 0.$$

For $p \geq 1$, define inductively

$$H\sigma = (\sigma - (-1)^{\dim(\sigma)} H\partial\sigma)\llcorner c(\sigma).$$

Proposition (3.2.1). *For all $n \geq 0$ we have the homotopy formula*

$$(\partial H_c - H_c\partial)\langle c(0), \cdots, c(n)\rangle$$
$$= (-1)^{n+1}\langle c(0), \cdots, c(n)\rangle + \sum \operatorname{sgn}(\sigma_0, \cdots, \sigma_n)\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle.$$

Here the σ_i run through all collections

$$\emptyset \subsetneqq \sigma_0 \subsetneqq \cdots \subsetneqq \sigma_n = \langle c(0), \cdots, c(\sigma_n)\rangle.$$

By definition, $\operatorname{sgn}(\sigma_0, \cdots, \sigma_n)$ is the signature of the permutation g of $\{0, \cdots, n\}$ with $g(i) =$ index of vertex in $\sigma_i - \sigma_{i-1}$.

Proof. Induction on n. When $n=0$, both sides of the equation are 0. For $n \geq 1$ we have

$$(\partial H - H\partial)\langle c(0), \cdots, c(n)\rangle$$
$$= (\partial\langle c(0), \cdots, c(n)\rangle + (-1)^{n-1}\partial H\partial\langle c(0), \cdots, c(n)\rangle)\llcorner c(0, \cdots, n)$$
$$+ (-1)^{n+1}(\langle c(0), \cdots, c(n)\rangle + (-1)^{n-1}H\partial\langle c(0), \cdots, c(n)\rangle)$$
$$- H\partial\langle c(0), \cdots, c(n)\rangle$$
$$= (-1)^{n+1}\langle c(0), \cdots, c(n)\rangle$$
$$+ (\partial\langle c(0), \cdots, c(n)\rangle$$
$$+ (-1)^{n-1}(\partial H - H\partial)\partial\langle c(0), \cdots, c(n)\rangle)\llcorner c(0, \cdots, n)$$
$$= (-1)^{n+1}\langle c(0), \cdots, c(n)\rangle$$
$$+ (-1)^{n-1}\sum\sum \operatorname{sgn}(\sigma_{i,0}, \cdots, \sigma_{i,n-1})$$
$$\cdot \langle c(\sigma_{i,0}), \cdots, c(\sigma_{i,n-1})\rangle\llcorner c(0, \cdots, n)$$
$$= (-1)^{n+1}\langle c(0), \cdots, c(n)\rangle + \sum \operatorname{sgn}(\sigma_0, \cdots, \sigma_n)\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle.$$

The double sum above runs over $0 \leq i \leq n$ and chains of simplices

$$\emptyset \subsetneqq \sigma_{i,0} \subsetneqq \cdots \subsetneqq \sigma_{i,n-1} = \langle c(0), \cdots, \widehat{c(i)}, \cdots, c(n)\rangle. \quad \text{Q.E.D.}$$

Corollary (3.2.2).

$$(-1)^{n+1}\langle c(0), \cdots, c(n)\rangle + \sum \operatorname{sgn}(\sigma_0, \cdots, \sigma_n)\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle$$
$$\equiv \partial H_c\langle c(0), \cdots, c(n)\rangle \quad \textit{mod simplices supported on } \partial\langle c(0), \cdots, c(n)\rangle.$$

Proof. The point is that $H\sigma$ is a formal linear combination of simplices with support in (i.e. having image contained in) σ. Thus $H\partial\langle c(0), \cdots, c(n)\rangle$ is supported on $\partial\langle c(0), \cdots, c(n)\rangle$. QED

Definition (3.2.3). With notation as above, the subdivision (with respect to the point c) of Δ^n is defined by

$$\text{subdivision}_c(\Delta^n) = \sum \operatorname{sgn}(\sigma_0, \cdots, \sigma_n)\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle.$$

We now consider the question of subdividing a product of simplices. Assume given points $c^{(i)} \in \Delta^{p_i}$ (sufficiently general so the above subdivision constructions are possible). Given

$$\sigma = \prod \sigma^{(i)} \subset \Delta^{p_1} \times \cdots \times \Delta^{p_r}$$

a product of simplices, we define

$$c(\sigma) = \prod c^{(i)}(\sigma^{(i)}) \in \Delta^{p_1} \times \cdots \times \Delta^{p_r}.$$

There is a natural notion of chains of products of simplices ($n = \sum p_i$)

$$\emptyset \subsetneqq \sigma_0 \subsetneqq \cdots \subsetneqq \sigma_n = \Delta^{p_1} \times \cdots \times \Delta^{p_r}$$

and maps

$$\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle : \Delta^n \to \Delta^{p_1} \times \cdots \times \Delta^{p_r}$$

using the linear structure of the product. In the same way, given $\phi : \Delta^m \to \prod \Delta^{p_i}$ and $E \in \prod \Delta^{p_i}$, we define

$$\phi\llcorner E : \Delta^{m+1} \to \prod \Delta^{p_i}.$$

The boundary, $\partial(\prod \Delta^{p_i})$, is a formal linear sum of maps of products of simplices into $\prod \Delta^{p_i}$, defined via the usual rules for boundaries of products. Finally, we define inductively

(3.2.4)

$$\text{subdivision}(\Delta^{p_1} \times \cdots \times \Delta^{p_r}) = (-1)^n \left(\partial_{\Pi\Delta} \circ \text{subdivision}\left(\partial\left(\prod \Delta^{p_i}\right)\right)\right)\llcorner c,$$

Here the basepoint for the subdivision of a given simplex σ is $c(\sigma)$. The notation "$\partial_{\Pi\Delta} \circ \cdots$" refers to composition with the map $\partial(\prod \Delta^{p_i}) \to \prod \Delta^{p_i}$. In words, the subdivision of the product is obtained by coning the subdivision of the boundary with vertex c a general point on the product.

Proposition (3.2.5). (i) *For $r = 1$ Definition* (3.2.4) *of subdivision coincides with the definition given in* (3.2.3).

(ii) *Viewing ∂ and subdivision as formal sums of maps, we have an identity of formal sums of maps $\Delta^{n-1} \to \prod \Delta^{p_i}$:*

$$\partial_{\Pi\Delta} \circ \text{subdivision}_{\partial(\Pi\Delta)} = \text{subdivision}_{\Pi\Delta} \circ \partial_{\Delta^n}.$$

Proof. For (i) note that $\sigma_n = \Delta^n$ so $c(\sigma_n) = c$ and

$$\sum \text{sgn}(\sigma_0, \cdots, \sigma_n)\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle$$
$$= \sum \text{sgn}(\sigma_0, \cdots, \sigma_n)\langle c(\sigma_0), \cdots, c(\sigma_{n-1})\rangle \llcorner c.$$

It is clear that $\partial_{\Delta^n} \circ \text{subdivision}(\partial\Delta^n) : \Delta^{n-1} \to \Delta^n$ is a linear combination of simplices $\langle c(\sigma_0), \cdots, c(\sigma_{n-1})\rangle$ with coefficients ± 1. Thus the simplices apearing in the expansions (3.2.3) and (3.2.4) are the same. We must check that they appear with the same sign. The "external" face of $\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle$ is $\langle c(\sigma_0), \cdots, c(\sigma_{n-1})\rangle$. If $c(i) \in \sigma_n - \sigma_{n-1}$ is the vertex not in σ_{n-1}, then this face appears in $\partial_i(\Delta^{n-1}) \subset \Delta^n$. It follows that $\langle c(\sigma_0), \cdots, c(\sigma_n)\rangle$ appears with sign $(-1)^{n+i}\text{sgn}(\sigma_0, \cdots, \sigma_{n-1})$ in (3.2.4). The desired assertion $\text{sgn}(\sigma_0, \cdots, \sigma_n) = (-1)^{n+i}\text{sgn}(\sigma_0, \cdots, \sigma_{n-1})$ is Lemma (3.1.3).

For (ii), note that by definition

$$\text{subdivision}_{\Pi\Delta} = (-1)^n(\partial_{\Pi\Delta} \circ \text{subdivision}_{\partial(\Pi\Delta)})\llcorner c.$$

The identity

$$(\varphi\llcorner) \circ \partial = (\varphi \circ \partial)\llcorner c - (-1)^p \varphi$$

for $\varphi : \Delta^p \to \prod \Delta^{p_i}$ implies by induction

$$\begin{aligned}
&\text{subdivision}_{\Pi\Delta} \circ \partial_{\Delta^n} \\
&= (-1)^n((\partial_{\Pi\Delta} \circ \text{subdivision}_{\partial(\Pi\Delta)}) \circ \partial_{\Delta^{n-1}})\llcorner c + \partial_{\Pi\Delta} \circ \text{subdivision}_{\partial(\Pi\Delta)} \\
&= -(\partial_{\Pi\Delta} \circ \partial_{\partial\Pi\Delta} \circ \text{subdivision}_{\partial\partial(\Pi\Delta)})\llcorner c + \partial_{\Pi\Delta} \circ \text{subdivision}_{\partial(\Pi\Delta)} \\
&= \partial_{\Pi\Delta} \circ \text{subdivision}_{\partial(\Pi\Delta)}. \quad \text{Q.E.D.}
\end{aligned}$$

(3.3) Subdivision for cubes. When $r = n$ and $p_1 = \cdots = p_n = 1$, subdivision becomes a linear sum of maps $\Delta^n \to \prod \Delta^1$. On the other hand, for each vertex $v \in \prod \Delta^1$ we have constructed in (1.3) a map $\pi_v : \Lambda_v^n \to \prod \Delta^1$. We will give another construction for the subdivision of $\prod \Delta^1$ by "subdividing" the Λ_v^n. Let $a = (a_1, \cdots, a_n) \in \Lambda^n$ with $\prod a_i \neq 0$. Define an action of $\mathcal{S}_n$ on Λ^n by

$$g(x_1, \cdots, x_n) = (a_1 a_{g^{-1}(1)}^{-1} x_{g^{-1}(1)}, \cdots, a_n a_{g^{-1}(n)}^{-1} x_{g^{-1}(n)}).$$

Define

$$a(\Lambda^i) = (0, \cdots, 0, a_{n-i+1}, \cdots, a_n),$$

so $a(\Lambda^0) = (0, \cdots, 0)$ and $a(\Lambda^1) = (0, \cdots, 0, a_n)$. Note

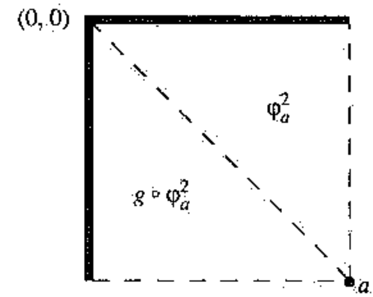

FIGURE 5

subdivision $(\Delta^1 \times \Delta^1)_a$

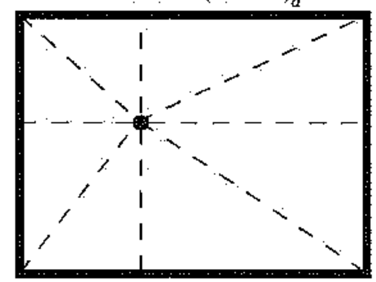

FIGURE 6

$$g(a(\Lambda^i))_j = \begin{cases} a_j & \text{if } j \notin \{g^{-1}(1), \cdots, g^{-1}(n-i)\}, \\ 0 & \text{otherwise}. \end{cases}$$

Define

$$\varphi_a^n : \Delta^n \to \Lambda^n,$$
$$\varphi_a^n(t_0, \cdots, t_n) = \sum t_i a(\Lambda^i),$$
$$\Psi^n = \sum_{\mathcal{S}_n} \text{sgn}(g) \cdot g \circ \varphi_a^n.$$

The picture for $n = 2$ is shown in Figure 5.

Let $c = (c^1, \cdots, c^n) \in \prod \Delta^1$ with $c^i \neq 0, 1$. For v a vertex, let $c_v = \pi_v^{-1}(c) \in \Lambda_v^n$. Recall we have defined in (3.1.1) a sign $\epsilon(v)$.

Proposition (3.3.1). *We have*

$$\text{subdivision}_{\Pi\Delta^1, c} = \sum_{v \text{ vertex}} \epsilon(v)\pi_v \circ \Psi_{c_v}^n.$$

This is straightforward except for the signs, and we leave it for the reader. The picture is shown in Figure 6.

(3.4) Subdivision and blowing up. Let $S_N \to \cdots \to S_1 \to S$ be an iterated tower of blowups of intersections of distinguished divisors with $S = \prod \Delta^1$ and distinguished divisors on S defined by setting coordinates $= 0, 1$. Note that the sum in (3.3.1) is over the vertices in S, but the expression makes perfect sense summed over vertices in S_p for any p. Our objective in this section is to show these expressions are in some sense homotopic independent of p. (Cf. (3.4.2) for a precise statement.)

Let $I \subset \{1, \cdots, n\}$ be a subset. Define

(3.4.1)
$$\Theta_I : \Lambda^n \times \Lambda^1 \to \Lambda^n; \quad \Theta_I(x_1, \cdots, x_n, t) = (m_1(x, t), \cdots, m_n(x, t))$$

where $m_i(x, t) = x_i t$ if $i \in I$ and $m_i = x_i$ otherwise. We will use Θ_I to pull back cycles. Note that the argument at the end of the proof of (2.1.1) applies in this case to show that if Z is a cycle on Λ^n meeting faces properly, then $\Theta_I^*(Z)$ on $\Lambda^n \times \Lambda^1$ meets faces properly also.

Let $\tilde{a} = (a, 1) \in \Lambda^n \times \Lambda^1$ and define

$$\Psi^{n+1} = \Psi^{n+1}_{\tilde{a}} : \Delta^{n+1} \to \Lambda^{n+1} = \Lambda^n \times \Lambda^1$$

(formal sum of maps). As usual we write $\partial_i : \Delta^n \hookrightarrow \Delta^{n+1}$ for the inclusion of the ith face. Given $f : \Delta^n \to S$, we write $\partial(f) = \sum(-1)^i f \circ \partial_i$. The basic homotopy statement we need is

Proposition (3.4.2).

$$\partial\left((-1)^{n+1} \sum_{\substack{v \in S_{N-1} \\ \text{vertex}}} \epsilon(v)\pi_v \circ \Theta_{I_v} \circ \Psi^{n+1}_{c_v}\right) = \sum_{\substack{v \in S_{N-1} \\ \text{vertex}}} \epsilon(v)\pi_v \circ \Psi^n_{c_v} + \sum_{\substack{v \in S_N \\ \text{vertex}}} \epsilon(v)\pi_v \circ \Psi^n_{c_v} + \mathscr{E}$$

where I_v is the set of coordinates in the v-coordinate system defining the face in S_{N-1} which is blown up in S_N, and $\mathscr{E}$ is a formal sum of maps $\Delta^n \to S$ whose image lands in the union of the distinguished divisors on S.

Proof. A sequence of lemmas:

Lemma (3.4.2.1). (i) *For $g \in \mathscr{S}_{n+1}$, the codimension-1 faces of $g \circ \varphi^{n+1}_{\tilde{a}}$ are*

$$g \circ \varphi^{n+1}_{\tilde{a}} \circ \partial_j = \operatorname{span}\langle g(\tilde{a}(\Lambda^0)), \cdots, \widehat{g(\tilde{a}(\Lambda^j))}, \cdots, g(\tilde{a}(\Lambda^{n+1}))\rangle.$$

(ii) *The image of $g \circ \varphi^{n+1}_{\tilde{a}} \circ \partial_0$ is defined by*

$$x_{g^{-1}(n+1)} = \tilde{a}_{g^{-1}(n+1)}.$$

(iii) *The image of $g \circ \varphi^{n+1}_{\tilde{a}} \circ \partial_{n+1}$ is defined by*

$$x_{g^{-1}(1)} = 0.$$

(iv) *For $j \neq 0, n+1$, let $h = (g^{-1}(n+1-j)\ g^{-1}(n+2-j)) \circ g$. (Here $(k\ell)$ is the transposition of k and ℓ.) Then $g \circ \varphi \circ \partial_j = h \circ \varphi \circ \partial_j$ (as maps, not simply as images). If $p \neq g$, h is a permutation, then $g \circ \varphi \circ \partial_j$ is not a face of $p \circ \varphi$.*

(v) $\Psi^{n+1}_{\tilde{a}} \circ \partial_j = 0$ *for $j \neq 0, n+1$.*

Proof. (i), (ii), and (iii) are clear. For example, the formula in (3.3) gives $g(\tilde{a}(\Lambda^i))_{g^{-1}(1)} = 0$ for $i \leq n$ so the span of these points is the hyperplane defined by $x_{g^{-1}(1)} = 0$.

To prove (iv), we remark that $g(\tilde{a}(\Lambda^i)) = p(\tilde{a}(\Lambda^i))$ for all $i \neq j$ means $\{g^{-1}(1), \cdots, g^{-1}(n+1-i)\} = \{p^{-1}(1), \cdots, p^{-1}(n+1-i)\}$ (unordered sets) for $i \neq j$. This implies $g^{-1}(\ell) = p^{-1}(\ell)$ for $\ell < n+1-j$ and $\ell > n-j+2$. Also

$$\{g^{-1}(n+1-j), g^{-1}(n-j+2)\} = \{p^{-1}(n+1-j), p^{-1}(n-j+2)\}.$$

This implies (iv).

Finally, (v) follows from (iv) since

$$\text{signature}(g) = -\text{signature}(h). \quad \text{Q.E.D.}$$

Lemma (3.4.2.2). *Let $\beta_i : \Lambda^n = \{x_i = \tilde{a}_i\} \subset \Lambda^{n+1}$ be the inclusion. Then*

$$\partial(\Theta_I \circ \Psi^{n+1}_{\tilde{a}}) = \sum_i (-1)^{n+1+i} \Theta_I \circ \beta_i \circ \Psi^n_{\tilde{a}(i)} + \mathscr{F}$$

where $\tilde{a}(i) = (\tilde{a}_1, \cdots, \widehat{\tilde{a}_i}, \cdots, \tilde{a}_{n+1})$ and $\mathscr{F}$ is a sum of maps $\Delta^n \to \Lambda^n$ with image in the union of divisors $\{x_i = 0\}$.

Proof. Up to sign, this follows from (3.4.2.1). To get the signs, one uses (3.1.3). Q.E.D.

Recall we have a tower $S_N \to \cdots \to S$. Let $\rho : S_N \to S_{N-1}$ be the last step. Let $Z \subset S_{N-1}$ be the center of the blowup ρ. Let $v \in Z \subset S_N$ be a vertex, and let $\pi_v : \Lambda^n_v \to S$ be the corresponding map. Let $I \subset \{1, \cdots, n\}$ be the set of variables defining Z near v, and let $p \notin I$ be given. Let $x_1, \cdots, x_n$ be the coordinates at v, and let ℓ be the edge through v defined by $x_j = 0$ for $j \neq p$. Note $\ell \subset Z$. Let $w \in \ell$ be the other vertex on ℓ (1.3.2), and write $y_1, \cdots, y_n$ for the coordinates at w. Let $J = \{j \mid y_j$ vanishes on $Z\}$, and let y_q be the y-coordinate not vanishing on ℓ at w. Let $c \in S$ be a point not lying on any distinguished divisor, and let c_v and c_w be the pre-images in Λ^n.

Lemma (3.4.2.3). *The components of*

$$\partial\left(\epsilon(v)\pi_v \circ \Theta_I \circ \Psi^{n+1}_{c_v} + (\epsilon(w)\pi_w \circ \Theta_J \circ \Psi^{n+1}_{c_w})\right)$$

supported on $\pi_v(\{x_p = c_{v,p}\}) = \pi_w(\{y_q = c_{w,q}\})$ *cancel.*

Proof. By (3.4.2.2) this amounts to the assertion that

$$(-1)^p \epsilon(v)\pi_v \circ \Theta_I \circ \beta_p \circ \Psi^n_{c_v} = (-1)^{q+1}\epsilon(w)\pi_w \circ \Theta_J \circ \beta_q \circ \Psi^n_{c_w}$$

as formal sums of maps $\Delta^n \to S$. Let $g = g(v, w) \in \mathcal{S}_n$, and let $\hat{g} \in \mathcal{S}_n$ be the permutation

$$\{1, \cdots, n\} \to \{1, \cdots, \hat{p}, \cdots, n+1\} \to \{1, \cdots, \hat{q}, \cdots, n+1\} \to \{1, \cdots, n\}$$

where the two outside maps are order preserving and the inner map is given by g extended by $n+1 \mapsto n+1$. Using (3.1.3) we can write the desired equality as

$$\pi_v \circ \Theta_I \circ \beta_p \circ \Psi^n_{c_v} = \text{signature}(\hat{g}) \circ \pi_w \circ \Theta_J \circ \beta_q \circ \Psi^n_{c_w}.$$

Let $\gamma : \Lambda^n \to \Lambda^n$ be the map corresponding to the permutation $\hat{g}$. The diagram

$$\begin{array}{ccc}
\Lambda^n & \xrightarrow{\gamma} & \Lambda^n \\
\downarrow{\scriptstyle \beta_p} & & \downarrow{\scriptstyle \beta_q} \\
\Lambda^{n+1} & & \Lambda^{n+1} \\
\downarrow{\scriptstyle \Theta_I} & & \downarrow{\scriptstyle \Theta_J} \\
\Lambda^n_v & & \Lambda^n_w \\
\downarrow{\scriptstyle \pi_v} & & \downarrow{\scriptstyle \pi_w} \\
S & = & S
\end{array}$$

commutes, and the lemma follows from

$$\gamma \circ \Psi^n = \text{signature}(\hat{g}) \circ \Psi^n,$$

which in turn is immediate from the definition of Ψ^n. Q.E.D.

Next, we consider terms

$$(-1)^{n+1+r}\pi_v \circ \Theta_I \circ \beta_r \circ \Psi^n$$

from the right-hand side of (3.4.2.2) with $r \in I$. (The terms with $r \notin I$ will cancel by (3.4.2.3).) Note that $r \in I$ corresponds in a natural way to $\nu \in S_N$ vertices lying over v. The composition

$$\Lambda^n \xrightarrow{\beta_r} \Lambda^{n+1} \xrightarrow{\Theta_I} \Lambda^n_v$$

coincides with the map (1.3.1)(ii) $\rho_{\nu,v} : \Lambda^n_\nu \to \Lambda^n_v$ up to a permutation of coordinates of the form

$$(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_{r-1}, z_n, z_r, \cdots, z_{n-1}).$$

By (3.1.4), this gives

$$(-1)^{n+1+r}\pi_v \circ \Theta_I \circ \beta_r \circ \Psi^n = (-1)^{n+1}\pi_v \circ \rho_{\nu,v} \circ \Psi^n = (-1)^{n+1}\pi_\nu \circ \Psi^n.$$

Finally, we note that $\Theta_I \circ \beta_{n+1} = \text{identity} : \Lambda^n \to \Lambda^n$.

Putting all this together, we find we have proved (3.4.2). Q.E.D.

4. Proof of the moving lemma

(4.1) Reductions. We work in the category of quasiprojective varieties over a field k. Let U be an open subvariety of a variety X. Let $\mathcal{C}_*$ be the cokernel

$$\mathcal{Z}(X, \cdot) \to \mathcal{Z}(U, \cdot) \to \mathcal{C}_* \to 0.$$

The moving lemma asserts that $\mathcal{C}_*$ is acyclic.

Lemma (4.1.1). *In the proof of the moving lemma, we may assume that* X *is projective.*

Proof. Consider the commutative diagram, with $\bar{X}$ = projective closure of X (to simplify we write $\mathcal{Z}(X)$ rather than $\mathcal{Z}(X, \cdot)$):

$$\begin{array}{ccccccccc}
0 \to & \mathcal{Z}(\bar{X}-U)/\mathcal{Z}(\bar{X}-X) & \to & \mathcal{Z}(\bar{X})/\mathcal{Z}(\bar{X}-X) & \to & \mathcal{Z}(\bar{X})/\mathcal{Z}(\bar{X}-U) & \to 0 \\
& \cap & & \cap & & \cap & \\
0 \to & \mathcal{Z}(X-U) & \to & \mathcal{Z}(X) & \to & \mathcal{Z}(X)/\mathcal{Z}(X-U) & \to 0
\end{array}$$

Assuming the moving lemma for projective X, the two left-hand vertical inclusions are quasi-isomorphisms of complexes. It follows that the right-hand vertical inclusion is also a quasi-isomorphism. But, again by the projective result, $\mathcal{Z}(\bar{X})/\mathcal{Z}(\bar{X}-U) \hookrightarrow \mathcal{Z}(U)$ is also a quasi-isomorphism. Q.E.D.

Lemma (4.1.2). *In the proof of the moving lemma, we may assume* k *is an infinite field.*

Proof. If k is finite, it has an infinite pro-ℓ extension k_ℓ for any ℓ. For $[k' : k] = d$ there are pullback and pushdown maps on cycles with composition multiplication by d. It follows that if $H_*(\mathscr{C}_{k_\ell,*}) = (0)$ then $H_*(\mathscr{C}_{k,*})$ is ℓ-torsion. Since ℓ is arbitrary, $H_*(\mathscr{C}_{k,*}) = (0)$. Q.E.D.

The following lemma is used to reduce to the case of cubes.

Lemma (4.1.3). *With notation as above, assume X is projective and k is an infinite field. Let $z \in \mathscr{Z}(U, n)$ be such that $\partial_i(z)$ lies in the image of $\mathscr{Z}(X, n-1) \to \mathscr{Z}(U, n-1)$ for $0 \le i \le n$. Let $\zeta \in H_*(\mathscr{C})$ be the class represented by z. Write $S = (\Delta^1)^n$ and for $c \in S(k)$ let $\Psi^n_c : \Delta^n \to S^n$ be the formal linear combination of maps defined in (3.3). For c fixed and sufficiently general (i.e., outside a finite union of closed subvarieties depending on z), there exists a cycle Z on $U \times S$ with the following properties:*

(i) Z meets all faces of $S \times U$ properly. For $\alpha : (\Delta^1)^{n-1} \hookrightarrow S$ defined by setting some coordinate $= 0, 1$, the closure of $\alpha^* Z$ on $(\Delta^1)^{n-1} \times X$ meets faces properly.

(ii) The cycle

$$\sum_{\substack{v \in S \\ \text{vertex}}} \epsilon(v) \Psi^n_c(Z)$$

is defined in $\mathscr{Z}(U, n)$. The image of this cycle in $\mathscr{C}$ is closed and represents the same class in $H_*(\mathscr{C})$ as ζ.

Proof. Define

$$\eta_n : \Delta^{n-1} \times \Delta^1 \to \Delta^n,$$
$$\eta_n((u_0, \cdots, u_p), (v_0, v_1)) = (u_0 v_0, \cdots, u_p v_0, v_1);$$
$$\psi_n : \Delta^n \times \Delta^1 \to \Delta^n,$$
$$\psi_n((y_0, \cdots, y_n), (x_0, x_1)) = (x_0 y_0, \cdots, x_0 y_{n-1}, x_1 + x_0 y_n).$$

The following identities are straightforward:

$$\psi_n \circ \partial_{0,\Delta^1} = \iota \underset{\text{def}}{=} \text{constant map to point } (0, \cdots, 0, 1) : \Delta^n \to \Delta^n,$$
$$\psi_n \circ \partial_{1,\Delta^1} = \text{identity} : \Delta^n \to \Delta^n,$$
$$\psi_n \circ \partial_{i,\Delta^n} = \partial_{i,\Delta^n} \circ \psi_{n-1} : \Delta^{n-1} \times \Delta^1 \to \Delta^n, \quad 0 \le i \le n-1,$$
$$\psi_n \circ \partial_{n,\Delta^n} = \eta_n : \Delta^{n-1} \times \Delta^1 \to \Delta^n.$$

From these, we deduce the identity in the free abelian group of (not necessarily linear) maps between products of simplices:

(4.1.3.1) $\partial_{\Delta^n} \circ \psi_{n-1} - \psi_n \circ \partial_{\Delta^n \times \Delta^1} = (-1)^n (\text{identity} - \iota + \eta_n - \partial_n \circ \psi_{n-1})$.

(Here $\partial_{\Delta^n \times \Delta^1}$ is the sum of maps $\Delta^{n-1} \times \Delta^1 \subset \Delta^n \times \Delta^1$ and $\Delta^n \subset \Delta^n \times \Delta^1$ with the usual signs.)

The same arguments as in the proof of (2.1.1) show that if z is a cycle on $U \times \Delta^n$ meeting faces properly, then the pullbacks $\eta_n^* z$ and $\psi_n^* z$ also meet faces properly. The terms $\partial_{\Delta^n} \circ \psi_{n-1}$, ι, and $\partial_n \circ \psi_{n-1}$ above factor through inclusions of faces $\Delta^{n-1} \subset \Delta^n$. Thus, if z has the property that the closure $\overline{\partial_i z}$ on $X \times \Delta^{n-1}$ meets faces properly for $0 \le i \le n$, we get from (4.1.3.1)

(4.1.3.2) $$\partial_{\Delta^n \times \Delta^1}(\psi_n^* z) \equiv (-1)^{n-1}(z + \eta_n^* z)$$

(modulo cycles whose closure over X meets faces properly).

Consider the diagram

(4.1.3.3)

$$\begin{array}{ccccccccc}
\Delta^2 \times (\Delta^1)^{n-1} & & \Delta^3 \times (\Delta^1)^{n-2} & & & & \Delta^n \times \Delta^1 & & \\
\cup & \searrow^{a_2} & \cup & \searrow \cdots & & & \cup & \searrow^{a_n} & \\
S = (\Delta^1)^n & \underset{b_2}{\longrightarrow} & \Delta^2 \times (\Delta^1)^{n-1} & \longrightarrow & \cdots & \longrightarrow & \Delta^{n-1} \times \Delta^1 & \underset{b_n}{\longrightarrow} & \Delta^n
\end{array}$$

where $a_i = \psi_i \times (\text{id})^{n-i}$ and $b_i = \eta_i \times (\text{id})^{n-i}$. Define

(4.1.3.4) $$Z = b_2^* \cdots b_n^* z$$

on $U \times S$. It follows from the definition of η_n that the maps b_i carry faces to faces; i.e. if $F \subset \Delta^r \times (\Delta^1)^{n-r}$ is a face, then

(4.1.3.5) $$b_n \circ b_{n-1} \circ \cdots \circ b_{r+1}(F) \subset \Delta^{n-1} \subset \Delta^n.$$

From this one sees easily that Z satisfies (i) in (4.1.3).

For the proof of (4.1.3)(ii), we find using (4.1.3.1)–(4.1.3.4)

(4.1.3.5) $$\sum_r \pm \partial_{\Delta^r \times (\Delta^1)^{n-r+1}}(a_r^* b_{r+1}^* \cdots b_n^* z) \equiv \pm z \pm Z$$

(mod cycles whose closure over X meets faces properly).

We now employ the subdivision maps $\Delta^n \to \prod \Delta^{p_i}$ defined in (3.2) to pullback cycles. Subdivision depends on the choice of a k-point $c = (c^{(1)}, \cdots, c^{(r)}) \in \prod \Delta^{p_i}$. It is easy to check that given a k-variety U and a cycle V on $U \times \prod \Delta^{p_i}$ meeting faces properly, there exists a closed algebraic subset $B \subsetneq \prod \Delta^{p_i}$ such that $c \notin B$ implies subdivision$_c^*(V)$ is defined and meets faces properly on $U \times \Delta^n$. In particular, when k is infinite as in our case, there exists such a c.

Using (3.2.5)(ii) and (4.1.3.5)

$$\sum_r \partial_{\Delta^{n+1}}(\text{subdivision}^*(a_r^* b_{r+1}^* \cdots b_n^* z)) \equiv \text{subdivision}^*(\pm z \pm Z)$$

(mod cycles whose closure over X meets faces properly).

By (3.2.2) we have in $\mathscr{Z}(U, n)$

$$\text{subdivision}^*(z) \equiv z \mod (\partial \mathscr{Z}(U, n+1) + \text{image}(\mathscr{Z}(X, n))).$$

By (3.3.1)

$$\text{subdivision}^*(Z) \equiv \sum_{\substack{v \in S \\ \text{vertex}}} \epsilon(v) \Psi_c^n(Z)$$

$$\mod(\partial \mathscr{Z}(U, n+1) + \text{image}(\mathscr{Z}(X, n))).$$

This completes the proof of (4.1.3). Q.E.D.

(4.2) End of the proof. We can now prove the moving lemma. We keep the notations of (4.1). Let ζ represent a class in $H_*(\mathscr{C})$, and let $z \in \mathscr{Z}(U, n)$ lift ζ. Let Z be a cycle on $U \times S$ satisfying (i) and (ii) of (4.1.3). Let

$$S_N \to \cdots \to S$$

be blowups of intersections of distinguished divisors so that for

$$\pi : \coprod_{\substack{v \in S_N \\ \text{vertex}}} \Lambda^n \to S$$

we have $\pi^! Z$ meets faces properly, where Z on $X \times S$ is the closure of Z. Since Z meets faces properly, it follows that $\pi^* Z$ has no components supported on faces properly contained in $U \times \coprod_v \Lambda^n$, so $\pi^! Z$ restricts to $\pi^* Z$ over U. By (4.1.3)(ii),

$$\sum_{\substack{v \in S \\ \text{vertex}}} \epsilon(v) \Psi_c^n(Z)$$

represents the class of ζ in $H_n(\mathscr{C})$. By (3.4.2) this class is the same as

$$\sum_{\substack{w \in S_N \\ \text{vertex}}} \epsilon(w) \Psi_c^n(\pi^* Z). \tag{4.2.1}$$

The class (4.2.1) is trivial because $\pi^* Z$ lifting to a cycle $\pi^! Z$ meeting faces properly implies (since the basepoint c is general) that (4.2.1) lifts to

$$\sum_{\substack{w \in S \\ \text{vertex}}} \epsilon(w) \Psi_c^n(\pi^! Z),$$

and $\zeta = 0$. QED

5. Specialization

(5.1) The cubical complex. In this section we show how the moving lemma gives rise to a specialization structure on the (candidate for the) category of mixed Tate motives constructed using algebraic cycles [2, 3]. Let k be a field. Write $\square^n = (\mathbb{P}^1 - \{1\})^n$ with *faces* defined by setting coordinates equal to 0 or ∞. The wreath product G of $\mathscr{S}_n$ with $\mathbf{Z}/2\mathbf{Z}^n$ acts on $\square^n$ by permuting and inverting coordinates. There is a natural representation $\text{Alt} : G \to \mathbf{Z}/2\mathbf{Z}$ which is the usual alternating representation on the symmetric group and is nontrivial on each factor of $\mathbf{Z}/2\mathbf{Z}$. We define $\mathscr{N}(r)^p$ to be the vector space of algebraic cycles of codimension r with $\mathbf{Q}$-coefficients on $\square^{2r-p}$ meeting all faces properly and alternating with respect to the action of G. One has a differential

$$d : \mathscr{N}(r)^p \to \mathscr{N}(r)^{p+1}$$

given by an alternating sum of restrictions to faces. (For more details cf. [3].) View the complex $\mathscr{Z}^r(k, \cdot)$ of codimension r cycles on simplices over k as a cohomological complex in negative degrees, so $\mathscr{Z}^r(k, p)$ is in degree $-p$. By [3, §4] one has a quasi-isomorphism

$$\mathscr{N}(r)^{\cdot} \approx \mathscr{Z}^r(k, \cdot)[-2r]. \tag{5.1.1}$$

There is also a product structure

$$\mathscr{N}(r)^p \times \mathscr{N}(s)^q \to \mathscr{N}(r+s)^{p+q}$$

obtained by composing the external product

$$\square^{2r-p} \times \square^{2s-q} \cong \square^{2(r+s)-(p+q)}$$

with alternating projection. In sum

$$\mathscr{N}^{\cdot} = \bigoplus_{r \geq 0} \mathscr{N}(r)^{\cdot}$$

has the structure of a graded differential graded algebra. (A word of warning: the codimension or Adams's grading on $\mathscr{N}^{\cdot}$ given by the index r in the previous formula does not have any effect on signs. The usual convention for graded objects would suggest doubling the r so all groups have even Adams's grading. We will not do this.)

One has (either [1, main theorem] which is now available since the moving lemma has been proved, or by [6])

$$H^p(\mathscr{N}(r)^{\cdot}) \cong \text{gr}_\gamma^r K_{2r-p}(k) \otimes \mathbf{Q}. \tag{5.1.2}$$

The candidate for the category of mixed Tate motives suggested in [3] is the category of finite-dimensional graded co-representations of the commutative Hopf algebra

(5.1.3) $$\chi_k = H^0(\mathrm{Bar}(\mathcal{N}^{\cdot})).$$

Here $\mathrm{Bar}(\mathcal{N}^{\cdot})$ is the bar construction on the algebra $\mathcal{N}^{\cdot}$. If $\mathcal{N}^{\cdot}$ is a $K(\pi, 1)$, i.e. if $H^*(\mathrm{Bar}(\mathcal{N}^{\cdot})) = (0)$ for $* \neq 0$, then this category will satisfy

$$\mathrm{Ext}^p(\mathbf{Q}, \mathbf{Q}(r)) \cong \mathrm{gr}^r_\gamma K_{2r-p}(k) \otimes \mathbf{Q}$$

as the Beilinson axioms require.

Let $\mathcal{O}$ be a discrete valuation ring with residue field k and quotient field F. We write $\mathcal{N}_{\mathcal{O}}$ for the analogous complex of cycles on cubes over $\mathrm{Spec}(\mathcal{O})$. Note $\mathcal{N}_{\mathcal{O}}$ is not an algebra, because the fibre product over $\mathrm{Spec}(\mathcal{O})$ of cycles meeting faces properly may not itself meet faces properly. (For example, one has $\mathcal{N}_k \subset \mathcal{N}_{\mathcal{O}}$, and the fibre product over $\mathrm{Spec}(O)$ of two cycles in $\mathcal{N}_k$ would not have the right codimension.) However, as a consequence of the "easy" moving lemma [1], $\mathcal{N}_{\mathcal{O}}$ does have a multiplication in the derived category. There is also a pullback map in the derived category

(5.1.4) $$i^*: \mathcal{N}_{\mathcal{O}} \to \mathcal{N}_k.$$

(5.2) Specialization for $\mathcal{N}$. Assume henceforth that $\mathcal{O}$ contains k as a subfield. (I would hope the discussion which follows is valid without this hypothesis.) The moving lemma and (5.1.1) yield a distinguished triangle

(5.2.1) $$\mathcal{N}_{\mathcal{O}} \xrightarrow{\alpha} \mathcal{N}_F \xrightarrow{\beta} \mathcal{N}_k[-1].$$

Choose a uniformizing parameter $\pi \in \mathcal{O}$, and let $\{\pi\} \in \mathcal{N}_F(1)^1$ be the 0-cycle. Define a map in the derived category

(5.2.2) $$\tau_\pi = \beta \circ (\text{multiplication by } \{\pi\}) : \mathcal{N}_F \to \mathcal{N}_k.$$

Let ξ be a variable with Adams's and cohomological degree 1 (so $\xi^2 = 0$). Define in the derived category

(5.2.3) $$T_\pi = \tau_\pi + (\text{mult. by } \xi) \circ \beta : \mathcal{N}_F \to \mathcal{N}_k[\xi].$$

Proposition (5.2.4). *Assuming $k \subset \mathcal{O}$, we have that T_π is compatible in the derived category with the algebra structures.*

Proof. Let $p^*: \mathcal{N}_k \to \mathcal{N}_{\mathcal{O}}$ be the pullback. Define

$$\theta_\pi = (\text{mult. by } \{\pi\}) \circ \alpha \circ p^* : \mathcal{N}_k[-1] \to \mathcal{N}_F.$$

One checks easily that

$$\beta \circ \theta_\pi = \text{identity},$$

so in the derived category we have a quasi-isomorphism

(5.2.4.1) $$\alpha \oplus \theta_\pi : \mathcal{N}_{\mathcal{O}} \oplus \mathcal{N}_k[-1] \to \mathcal{N}_F.$$

Also, by using the easy moving lemma to choose cycles on $\square^n_{\mathcal{O}}$ meeting all faces of $\square^n_k$ properly, one gets

$$\tau_\pi \circ \alpha = i^* : \mathcal{N}_{\mathcal{O}} \to \mathcal{N}_k,$$

since $\{\pi\}^2 = 0$, $\tau_\pi \circ \theta_\pi = 0$, so the assertion of the proposition reduces to two claims: (i) $\tau_\pi : \mathcal{N}_F \to \mathcal{N}_k$ is compatible with the algebra structure. (ii) $\beta : \mathcal{N}_F \to \mathcal{N}_k[-1]$ satisfies the identity

$$\beta(x \cdot y) = \tau_\pi(x) \cdot \beta(y) + (-1)^{\deg(y)} \beta(x) \cdot \tau_\pi(y).$$

(Of course, this identity and those that follow below are to hold in the derived category, and it is abusive to insert elements. The formulas should be interpreted as commutative diagrams of arrows.) Claim (i) reduces by (5.2.4.1) to the easily verified equalities

$$\tau_\pi(\alpha(x) \cdot \alpha(y)) = \tau_\pi \alpha(x \cdot y) = i^*(x \cdot y) = i^*(x) \cdot i^*(y) = \tau_\pi(\alpha(x)) \cdot \tau_\pi(\alpha(y)),$$
$$\tau_\pi(\alpha(x) \cdot \theta_\pi(y)) = 0 = \tau_\pi(\alpha(x)) \cdot \tau_\pi(\theta_\pi(y)).$$

Claim (ii) follows from the identity

$$\beta(\alpha(x) \cdot \theta_\pi(z)) = i^*(x) \cdot z = \tau_\pi \alpha(x) \cdot \beta(\theta_\pi(z))$$

which is checked directly on cycles. Q.E.D.

(5.3) Mixed motives over $\mathrm{Spec}(\mathcal{O})$. We continue to assume $k \subset \mathcal{O}$. Let χ_F and χ_k be as in (5.1.3). The following two results are corollaries of (5.2.4).

Proposition (5.3.1). *Let t be a variable placed in Adams's degree 1 which is primitive for comultiplication. Then T_π induces a homomorphism of graded Hopf algebras*

$$T_\pi : \chi_F \to \chi_k[t].$$

Proof. The point is that maps in the derived category which are compatible with the multiplication structure induce maps in the derived category on the bar complex. Q.E.D.

Let $\chi^+ \subset \chi$ be the augmentation ideal, and write $\mathcal{M} = \chi^+/(\chi^+)^2$ for the indecomposibles. Write $\mathcal{L}$ for the pro-object dual to $\mathcal{M}$ (so the dual of $\mathcal{L}$ is $\mathcal{M}$). One has $\mathcal{M}_k = \mathcal{M}_{k,1} \oplus \mathcal{M}_{k,2} \oplus \cdots$, and $\mathcal{M}_{k,1} = k^\times \otimes \mathbf{Q}$ [3].

In particular, the valuation map $F^\times \to \mathbf{Z}$ dualizes to a homomorphism of graded Lie algebras

$$\mathrm{val}:\mathbf{Q}_{-1}\to \mathscr{L}_F.$$

Proposition (5.3.2). *Assume* $k\subset\mathscr{O}$. *Then, with the above notation, one has a specialization map*

$$\mathrm{sp}_\pi:\mathscr{L}_k\to\mathscr{L}_F.$$

The image of sp_π *centralizes the image of* val.

Proof. Straightforward from (5.3.1). Q.E.D.

As mentioned in the introduction, if we write $\Psi_\pi(M)$ for the $\mathscr{L}_k$-representation coming via sp_π from a finite graded representation M of $\mathscr{L}_F$, then val(1) induces a map of mixed Tate motives over Spec(k)

$$N:\Psi_\pi(M)\to\Psi_\pi(M)(-1). \tag{5.3.3}$$

Motivated by the construction of e.g. étale sheaves on Spec($\mathscr{O}$), we can formulate the following candidate for the category of mixed Tate motives over Spec($\mathscr{O}$)

Definition (5.3.4). The category $\mathscr{MTM}$(Spec($\mathscr{O}$)) has as objects triples

$$(M_F, M_k, w)$$

where M_F (resp. M_k) is a representation of $\mathscr{L}_F$ (resp. $\mathscr{L}_k$) and $w: M_k\to\ker N\subset\Psi_\pi(M)$ is a map of $\mathscr{L}_k$-modules.

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