# Zeta and $L$ Functions

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The purpose of this lecture is to give the general properties of zeta functions and Artin's $L$ functions in the setting of *schemes*. I will restrict myself mainly to the formal side of the theory; for the connection with $l$-adic cohomology and Lefschetz's formula, see Tate's lecture.

## §1. ZETA FUNCTIONS

### 1.1. DIMENSION OF SCHEMES

All schemes considered below are supposed to be *of finite type over* **Z**. Such a scheme $X$ has a well-defined *dimension* denoted by $\dim X$. It is the maximum length $n$ of a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n, \qquad Z_i \neq Z_{i+1}$$

of closed irreducible subspaces of $X$. If $X$ itself is irreducible, with generic point $x$, and if $k(x)$ is the corresponding residue field, one has:

$$\dim X = \text{Kronecker dimension of } k(x). \tag{1}$$

(The Kronecker dimension of a field $E$ is the transcendence degree of $E$ over the prime field, augmented by 1 if $\text{char} E = 0$.)

### 1.2. CLOSED POINTS

Let $X$ be a scheme and let $x \in X$. The following properties are equivalent:

a. $\{x\}$ is closed in $X$.

b. The residue field $k(x)$ is finite.

The set of closed points of $X$ will be denoted by $\bar{X}$; we view it as a discrete topological space, equipped with the sheaf of fields $k(x)$; we call $\bar{X}$ the *atomization* of $X$. If $x \in \bar{X}$, the *norm* $N(x)$ of $x$ is the number of elements of $k(x)$.

### 1.3. ZETA FUNCTIONS

The zeta function of a scheme $X$ is defined by the eulerian product

$$\zeta(X, s) = \prod_{x \in \bar{X}} \frac{1}{1 - 1/N(x)^s}. \tag{2}$$

It is easily seen that there are only a finite number of $x \in \bar{X}$ with a given norm. This is enough to show that the above product is a *formal Dirichlet series* $\Sigma a_n/n^s$, with integral coefficients. In fact, that series converges, as the following theorem shows:

**Theorem 1.** The product $\zeta(X, s)$ converges absolutely for

$$R(s) > \dim X.$$

(As usual, $R(s)$ denotes the *real part* of $s$.)

**Lemma.** (a) Let $X$ be a finite union of schemes $X_i$. If Theorem 1 is valid for each of the $X_i$'s, it is valid for $X$. (b) If $X \to Y$ is a finite morphism, and if Theorem 1 is valid for $Y$, it is valid for $X$.

Using this lemma (which is elementary) and induction on dimension, one reduces Theorem 1 to the case

$$X = \text{Spec } A[T_1, \ldots, T_n],$$

where the ring $A$ is either $\mathbf{Z}$ or $\mathbf{F}_p$. In the first case, $\dim X = n + 1$, and the product (2) gives (after collecting some terms together):

$$\zeta(X, s) = \prod_p \frac{1}{1 - p^{n-s}} = \zeta(s - n).$$

In the second case, $\dim X = n$, and $\zeta(X, s) = 1/(1 - p^{n-s})$. In both cases, we have absolute convergence for $R(s) > \dim X$.

## 1.4. ANALYTIC CONTINUATION OF ZETA FUNCTIONS

One *conjectures* that $\zeta(X, s)$ can be continued as a meromorphic function in the entire $s$-plane; this, at least, has been proved for many schemes. However, in the general case, one knows only the following much weaker:

**Theorem 2.** $\zeta(X, s)$ can be continued analytically (as a meromorphic function) in the half-plane $R(s) > \dim X - \frac{1}{2}$.

The singularities of $\zeta(X, s)$ in the strip

$$\dim X - \tfrac{1}{2} < R(s) \leqq \dim X$$

are as follows:

**Theorem 3.** Assume $X$ to be irreducible, and let $E$ be the residue field of its generic point.

a. If $\operatorname{char} E = 0$, the only pole of $\zeta(X, s)$ in $R(s) > \dim X - \frac{1}{2}$ is $s = \dim X$, and it is a simple pole.

b. If $\operatorname{char} E = p \neq 0$, let $q$ be the highest power of $p$ such that $E$ contains the field $\mathbf{F}_q$. The only poles of $\zeta(X, s)$ in $R(s) > \dim X - \frac{1}{2}$ are the points

$$s = \dim X + \frac{2\pi i n}{\log(q)}, \qquad n \in \mathbf{Z},$$

and they are simple poles.

**Corollary 1.** For any nonempty scheme $X$, the point $s = \dim X$ is a pole of $\zeta(X, s)$. Its order is equal to the number of irreducible components of $X$ of dimension equal to $\dim X$.

**Corollary 2.** The domain of convergence of the Dirichlet series $\zeta(X, s)$ is the half-plane $R(s) > \dim X$.

Theorem 2 and Theorem 3 are deeper than Theorem 1. Their proof uses the "Riemann hypothesis for curves" of Weil [7] combined with the technique of "fibering by curves" (i.e., maps $X \to Y$ whose fibers are of dimension 1). One may also deduce them from the estimates of Lang-Weil [5] and Nisnevič [6].

### 1.5. SOME PROPERTIES AND EXAMPLES

$\zeta(X, s)$ depends only on the *atomization* $\bar{X}$ of $X$. In particular, it does not change by radicial morphism, and we have

$$\zeta(X_{\mathrm{red}}, s) = \zeta(X, s). \tag{3}$$

If $X$ is a disjoint union (which may be infinite) of subschemes $X_i$, we have

$$\zeta(X, s) = \Pi\, \zeta(X_i, s),$$

with absolute convergence for $R(s) > \dim X$. It is even enough that $\bar{X}$ be the disjoint union of the $\bar{X}_i$'s. For instance, if $f : X \to Y$ is a morphism, we may take for $X_i$'s the fibers $X_y = f^{-1}(y)$, $y \in \bar{Y}$, and we get:

$$\zeta(X, s) = \prod_{y \in \bar{Y}} \zeta(X_y, s). \tag{4}$$

(This, with $Y = \mathrm{Spec}(\mathbf{Z})$, was the original definition of Hasse-Weil.) Note that the $X_y$'s are schemes over the finite fields $k(y)$; that is, they are "algebraic varieties."

If $X = \mathrm{Spec}(A)$, where $A$ is the ring of integers of a number field $K$ $\zeta(X, s)$ coincides with the classical zeta function $\zeta_K$ attached to $K$. For $A = \mathbf{Z}$, we get Riemann's zeta.

If $\mathbf{A}^n(X)$ is the affine $n$-space over a scheme $X$, we have

$$\zeta(\mathbf{A}^n(X), s) = \zeta(X, s - n).$$

Similarly,

$$\zeta(\mathbf{P}^n(X), s) = \prod_{m=0}^{m=n} \zeta(X, s - m).$$

### 1.6. SCHEMES OVER A FINITE FIELD

Let $X$ be a scheme over $\mathbf{F}_q$. If $x \in \bar{X}$, the residue field $k(x)$ is a finite extension of $\mathbf{F}_q$; let $\deg(x)$ be its degree. We have

$$N(x) = q^{\deg(x)},$$

and

$$\zeta(X, s) = Z(X, q^{-s}) \tag{5}$$

where $Z(X, t)$ is the power series defined by the product:

$$Z(X, t) = \prod_{x \in \bar{X}} \frac{1}{1 - t^{\deg(x)}}. \tag{6}$$

The product (6) converges for $|t| < q^{-\dim X}$.

**Theorem 4 (Dwork).** $Z(X, t)$ is a rational function of $t$.

(See[3] for the proof.)

In particular, $\zeta(X, s)$ is meromorphic in the whole plane and periodic of period $2\pi i/\log(q)$.

There is another expression of $Z(X, t)$ which is useful:

Let $k = \mathbf{F}_q$, and denote by $k_n$ the extension of $k$ with degree $n$. Let $X_n = X(k_n)$ be the set of points of $X$ with value in $k_n/k$. Such a point $P$ can be viewed as a *pair* $(x, f)$, with $x \in \bar{X}$, and where $f$ is a $k$-isomorphism of $k(x)$ into $k_n$. We have

$$\bigcup X_n = X(\bar{k}),$$

where $\bar{k}$ is the algebraic closure of $k$.

It is easily seen that the $X_n$'s are *finite*. If we put:

$$\nu_n = \mathrm{Card}(X_n),$$

we have

$$\log Z(X, t) = \sum_{n=1}^{\infty} \frac{\nu_n t^n}{n}. \tag{7}$$

### 1.7. FROBENIUS

We keep the notations of 1.6. Let $F : X \to X$ be the "Frobenius morphism" of $X$ into itself (i.e., $F$ is the identity on the topological space $X$, and it acts on the sheaf $O_X$ by $\varphi \mapsto \varphi^q$). If we make $F$ operate on $X(\bar{k})$, the *fixed points* of the $n$th iterate $F^n$ of $F$ are *the elements of* $X_n$. In particular, the number $\nu_n$ is *the number* $\Lambda(F^n)$ *of fixed points of* $F^n$. This remark, first made by Weil, is the starting point of his interpretation of $\nu_n$ as a *trace*, in Lefschetz's style.

## §2. *L* FUNCTIONS

### 2.1 FINITE GROUPS ACTING ON A SCHEME

Let $X$ be a scheme, let $G$ be a finite group, and suppose that $G$ acts on $X$ on the right; we also assume that the quotient $X/G = Y$

exists (i.e., $X$ is a union of affine open sets which are stable by $G$). The atomization $\bar{Y}$ of $Y$ may be identified with $\bar{X}/G$. More precisely, let $x \in \bar{X}$, let $y$ be its image in $\bar{Y}$, and let $D(x)$ be the corresponding decomposition subgroup; we have $g \in D(x)$ if and only if $g$ leaves $x$ fixed. There is a natural epimorphism

$$D(x) \to \operatorname{Gal} k(x)/k(y).$$

Its kernel $I(x)$ is called the *inertia subgroup* corresponding to $x$; when $I(x) = \{1\}$, the morphism $X \to Y$ is *étale* at $x$.

Since $D(x)/I(x)$ can be identified with $\operatorname{Gal}(k(x)/k(y))$, it is a cyclic group, with a canonical generator $F_x$, called the *Frobenius element* of $x$.

## 2.2. ARTIN'S DEFINITION OF $L$ FUNCTIONS

Let $\chi$ be a character of $G$ (i.e. a linear combination, with coefficients in $\mathbf{Z}$, of irreducible complex characters). For each $y \in \bar{Y}$, and for each integer $n$, let $\chi(y^n)$ be the mean value of $\chi$ on the $n$th power $F_x^n$ of the Frobenius element $F_x \in D(x)/I(x)$, where $x \in \bar{X}$ is any lifting of $y$. Artin's definition of the $L$ function $L(X, \chi; s)$ is the following (cf. [1]):

$$\log L(X, \chi; s) = \sum_{y \in \bar{Y}} \sum_{n=1}^{\infty} \frac{\chi(y^n) N(y)^{-ns}}{n}. \tag{8}$$

When $\chi$ is the character of a linear representation $g \mapsto M(g)$, we have

$$L(X, \chi; s) = \prod_{y \in \bar{Y}} \frac{1}{\det(1 - M(F_x)/N(y)^s)}, \tag{9}$$

where $M(F_x)$ is again defined as the mean value of $M(g)$, for $g \mapsto F_x$.

Both expressions (8) and (9) converge absolutely when

$$R(s) > \dim X.$$

## 2.3. FORMAL PROPERTIES OF THE $L$ FUNCTIONS

a. $L(X, \chi)$ depends on $X$ only through its atomization $\bar{X}$.

b. $L(X, \chi + \chi') = L(X, \chi). L(X, \chi')$.

c. If $\bar{X}$ is the disjoint union of the $\bar{X}_i$'s, with $X_i$ stable by $G$ for each $i$, we have

$$L(X, \chi; s) = \prod L(X_i, \chi; s)$$

with absolute convergence for $R(s) > \dim X$.

d. Let $\pi : G \to G'$ be a homomorphism, and let $\pi_* X = X \times^G G'$ be the scheme deduced from $X$ by "extension of the structural group." Let $\chi'$ be a character of $G'$, and let $\pi^*\chi' = \chi' \circ \pi$ be the corresponding character of $G$. We have

$$L(X, \pi^*\chi') = L(\pi_* X, \chi'). \tag{10}$$

e. Let $\pi : G' \to G$ be a homomorphism, and let $\pi^* X$ denote the scheme $X$ on which $G'$ operates through $\pi$. Let $\chi'$ be a character of $G'$, and let $\pi_* \chi'$ be its direct image, which is a character of $G$ (when $G'$ is a subgroup of $G$, $\pi_* \chi'$ is the "induced character" of $\chi'$). We have

$$L(X, \pi_* \chi') = L(\pi^* X, \chi'). \tag{11}$$

f. Let $X = \mathrm{Spec}(\mathbf{F}_{q^n})$, $Y = \mathrm{Spec}(\mathbf{F}_q)$, $G = \mathrm{Gal}(\mathbf{F}_{q^n}/\mathbf{F}_q)$, and $\chi$ an irreducible character of $G$. We have

$$L(X, \chi; s) = \frac{1}{1 - \chi(F) q^{-s}}, \tag{12}$$

where $F$ is the Frobenius element of $G$.

It is not hard to see that *properties* (a) *to* (f) *uniquely characterize the L functions.*

g. If $\chi = 1$ (unit character), $L(X, 1) = \zeta(X/G)$.

h. If $\chi = r$ (character of the regular representation), we have

$$L(X, r) = \zeta(X).$$

By combining (h) and (b), one gets the following formula (which is one of the main reasons for introducing $L$ functions):

$$\zeta(X) = \prod_{\chi \in \mathrm{Irr}(G)} L(X, \chi)^{\deg(\chi)}, \tag{13}$$

where $\mathrm{Irr}(G)$ denotes the set of *irreducible characters* of $G$, and $\deg(\chi) = \chi(1)$.

There is an analogous result for $\zeta(X/H)$, when $H$ is a subgroup of $G$; one replaces the regular representation by the permutation representation of $G/H$.

### 2.4. SCHEMES OVER A FINITE FIELD

Let $X$ be an $\mathbf{F}_q$-scheme and assume that the operations of $G$ are $\mathbf{F}_q$-automorphisms of $X$. The scheme $Y = X/G$ is then also an $\mathbf{F}_q$-scheme.

On the set $X(\bar{k})$, we have two kinds of operators: the Frobenius endomorphism $F$ (cf. 1.7) and the automorphisms defined by the elements of $G$; if $g \in G$, we have $F \circ g = g \circ F$.

If we put as usual $t = q^{-s}$, we can transform $L(X, \chi; s)$ into a function $L(X, \chi; t)$ of $t$. An elementary calculation gives:

$$\log L(X, \chi; t) = \sum_{n=1}^{\infty} \nu_n(\chi) t^n/n, \tag{14}$$

with

$$\nu_n(\chi) = \frac{1}{(G)} \sum_{g \in G} \chi(g^{-1}) \Lambda(gF^n), \tag{15}$$

where $(G) = \mathrm{Card}(G)$, and $\Lambda(gF^n)$ is the number of fixed points of $gF^n$ (acting on $X(\bar{k})$).

(These formulae could have been used to *define* the $L$ functions; they make the verification of properties (a) to (f) very easy.)

*Remark.* It is not yet known that $L(X, \chi; t)$ is a *rational function* of $t$. However, this is true in the following special cases:

a. When $X$ is projective and smooth over $\mathbf{F}_q$: this follows from $l$-adic cohomology (Artin-Grothendieck).
b. When Artin-Schreier or Kummer theory applies; that is, when $G$ is cyclic of order $p^N$, or of order $m$ prime to $p$, with $m$ dividing $q - 1$. This can be proved by Dwork's method; the case $G = \mathbf{Z}/p\mathbf{Z}$ has been studied in some detail by Bombieri.

(Added in proof: The rationality of the $L$ functions has now been proved by Grothendieck. See his Bourbaki's lecture, n° 279.)

### 2.5. ARTIN-SCHREIER EXTENSIONS

It would be easy—but too long—to give various examples of $L$ functions, in particular for an abelian group $G$. I will limit myself to one such example:

Let $Y$ be an $\mathbf{F}_q$-scheme, and let $a$ be a section of the sheaf $O_Y$. In the affine line $Y[T]$, let $X$ be the closed subscheme defined by the equation

$$T^p - T = a.$$

If we put $G = \mathbf{Z}/p\mathbf{Z}$, the group $G$ acts on $X$ by $T \mapsto T + 1$, and

$X/G = Y$; we get in this way an *étale covering*. Let $w$ be a primitive $p$th root of unity in $\mathbf{C}$, and let $\chi$ be the character of $G$ defined by $\chi(n) = w^n$. The $L$ function $L(X, \chi; t)$ is given by formula (14); its coefficients $\nu_n(\chi)$ can be written here in the following form:

$$\nu_n(\chi) = \sum_{y \in Y_n} w^{\mathrm{Tr}_n a(y)}, \tag{16}$$

where $Y_n = Y(k_n)$, and $\mathrm{Tr}_n$ is the trace map from $k_n = \mathbf{F}_{q^n}$ to $\mathbf{F}_p$. The above expression is a typical "exponential sum." If, for instance, we take for $Y$ the multiplicative group $\mathbf{G}_m$, and put $a = \lambda y + \mu y^{-1}$, we get the so-called "Kloosterman sums." This connection between $L$ functions and exponential sums was first noticed by Davenport-Hasse [2] and then used by Weil [8] to give estimates in the one-dimensional case.

## 2.6. ANALYTIC CONTINUATION OF $L$ FUNCTIONS

Theorems 2 and 3 have analogues for $L$ functions. First:

**Theorem 5.** $L(X, \chi; s)$ can be continued analytically (as a meromorphic function) in the half-plane $R(s) > \dim X - \frac{1}{2}$.

The singularities of $L(X, \chi; s)$ in the critical strip

$$\dim X - \tfrac{1}{2} < R(s) \leq \dim X$$

can be determined, or rather reduced to the classical case $\dim X = 1$. We use the following variant of the "fibering by curves" method:

**Lemma.** Let $f : X \to X'$ be a morphism which commutes with the action of the group $G$. Assume that all geometric fibers of $f$ are irreducible curves. Then

$$L(X, \chi; s) = H(s) \cdot L(X', \chi; s - 1), \tag{17}$$

where $H(s)$ is holomorphic and $\neq 0$ for $R(s) > \dim X - \frac{1}{2}$.

This lemma gives a reduction process to dimension 1 (and even to dimension 0 if $X$ is a scheme over a finite field). The result obtained in this way is a bit involved, and I will just state a special case:

**Theorem 6.** Assume that $X$ is irreducible, and that $G$ operates faithfully on the residue field $E$ of the generic point of $X$. Let $\chi$ be a character of $G$, and let $\langle\chi, 1\rangle$ be the multiplicity of the identity character 1 in $\chi$. The order of $L(X, \chi)$ at $s = \dim X$ is equal to $-\langle\chi, 1\rangle$.

**Corollary.** If $\chi$ is a non-trivial irreducible character, $L(X, \chi)$ is holomorphic and $\neq 0$ at the point $s = \dim X$.

### 2.7. ARTIN-ČEBOTAREV'S DENSITY THEOREM

Let $Y$ be an irreducible scheme of dimension $n \geqq 1$. By using the fact that $\zeta(Y, s)$ has a simple pole at $s = n$, we get easily:

$$\sum_{y \in \bar{Y}} \frac{1}{N(y)^s} \sim \log \frac{1}{s - n} \qquad \text{for } s \to n. \tag{18}$$

A subset $M$ of $\bar{Y}$ has a *Dirichlet density* $m$ if we have

$$\left(\sum_{y \in M} \frac{1}{N(y)^s}\right) \Big/ \log \frac{1}{s-n} \to m \qquad \text{for } s \to n. \tag{19}$$

For $Y = \mathrm{Spec}(\mathbf{Z})$, this is the usual definition of the Dirichlet density of a set of prime numbers.

Now let $X$ verify the assumptions of Theorem 6, and let $Y = X/G$. Assume that $\dim X \geqq 1$ and that $G$ operates freely (i.e., $I(x) = \{1\}$ for all $x \in \bar{X}$). If $y \in \bar{Y}$, the Frobenius element $F_x$ of a corresponding point $x \in \bar{X}$ is a well defined element of $\bar{G}$, and its conjugation class $F_y$ depends only on $y$.

**Theorem 7.** Let $R \subset G$ be a subset of $G$ stable by conjugation. The set $\bar{Y}_R$ of elements $y \in \bar{Y}$ such that $F_y \subset R$ has Dirichlet density equal to $\mathrm{Card}(R)/\mathrm{Card}(G)$.

This follows by standard arguments from the corollary to Theorem 6.

**Corollary.** $\bar{Y}_R$ is infinite if $R \neq \emptyset$.

*Remark.* A slightly more precise result has been obtained by Lang [4] for "geometric" coverings and also for coverings obtained by extension of the ground field.

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